10.1. Unique solution

Let $k > 0$ be a positive constant. Let D be a bounded planar domain in $\mathbb{R}^2$ (that is, $(x, y) \in D \subset \mathbb{R}^2$), and let $g = g(x, y)$ be a continuous function defined on the boundary ∂D (that is, $g \in C(\partial\Omega)$. Let $u = u(x, y)$ be a solution to the Dirichlet problem for the reduced Helmholtz energy in D. That is, let u solve

$$\begin{cases} \Delta u(x,y) - ku(x,y) &= 0, & \text{for } (x,y) \in D, \\ u(x,y) &= g(x,y), & \text{for } (x,y) \in \partial D. \end{cases}$$

Show that there exists at most a unique solution twice differentiable in D and continuous in $\overline{D}$, that is, $u \in C^2(D) \cap C(\overline{D})$.

Hint: Assume that there exist two solutions u_1 and u_2, and consider the difference $v = u_1 - u_2$. Arguing as in the proof of the weak maximum principle for the Laplace equation (Theorem 7.5 and Remark 7.6 from Pinchover's book), show that $\max_{\overline{D}} v = \max_{\partial D} v$ and $\min_{\overline{D}} v = \min_{\partial D} v$. Then, use this information to conclude that $u_1 = u_2$.

10.2. The mean-value principle Let D be a planar domain, and let $B_R((x_\circ, y_\circ))$ (ball of radius R centered at $(x_\circ, y_\circ)$) be fully contained in D. Let u be an harmonic function in D, $\Delta u = 0$ in D. Then, the mean-value principle says that the value of u at $(x_\circ, y_\circ)$ is the average value of u on $\partial B_R((x_\circ, y_\circ))$. That is,

$$u(x_\circ, y_\circ) = \frac{1}{2\pi R} \oint_{\partial B_R((x_\circ,y_\circ))} u(x(s), y(s))\, ds = \frac{1}{2\pi} \int_0^{2\pi} u(x_\circ + R\cos\theta, y_\circ + R\sin\theta)\, d\theta.$$

Show that $u(x_\circ, y_\circ)$ is also equal to the average of u in $B_R((x_\circ, y_\circ))$, that is,

$$u(x_\circ, y_\circ) = \frac{1}{\pi R^2} \int_{B_R((x_\circ,y_\circ))} u(x,y)\, dx\, dy.$$

10.3. Weak maximum principle Let B_1 denote the unit ball in $\mathbb{R}^2$ centered at the origin, and let $u = u(x, y)$ be twice differentiable in B_1 and continuous in $\overline{B_1}$. Suppose that u solves the Dirichlet problem

$$\begin{cases} \Delta u(x,y) &= -1, & \text{for } (x,y) \in B_1, \\ u(x,y) &= g(x,y), & \text{for } (x,y) \in \partial B_1. \end{cases}$$

Show that

$$\max_{\bar{B}_1} u \le \frac{1}{2} + \max_{\partial B_1} g.$$

Hint: search for a simple *function w such that $\Delta w = 1$, and use it to reduce the problem to an application of the weak maximum principle for harmonic functions.*

10.4. Multiple Choice Determine the correct answer.

(a) Consider the Neumann problem for the Poisson equation

$$\begin{cases} \Delta u = \rho, & \text{in } D, \\ \partial_\nu u = g, & \text{on } \partial D, \end{cases}$$

where $D = B(0, R)$ is the ball of radius $R > 0$ with centre in the origin of $\mathbb{R}^2$, and ρ and g are given in polar coordinates (r, θ) by

$$\rho(r,\theta) = r^\alpha \sin^2(\theta), \text{ and } g(r,\theta) = C\cos^2(\theta) + r^{2021}\sin(\theta),$$

for some constants $\alpha > 0$ and $C > 0$. For which values of $C > 0$ does the problem satisfy the Neumann's *necessary* condition for existence of solutions?

- ☐ $C = \frac{R^{\alpha+1}}{\alpha+2}$
- ☐ $C = \frac{R^{\alpha+1}}{\alpha+1}$
- ☐ $C = \frac{R^{\alpha+2}}{\alpha+2}$