1. Multiple choice questions.

Cross the *unique* correct answer.

(a) State the order and the linearity or type of nonlinearity for the following PDE: $(u_{xx} + 7u_y)^5 = u_x u_y + u$.

- ☐ Second order, nonlinear. It is quasilinear.
- ☐ Second order, nonlinear. It is not quasilinear.
- ☐ First order, linear.
- ☐ First order, nonlinear. It is not quasilinear.

(b) State the order and the linearity or type of nonlinearity for the following PDE: $\cosh(x^2 - y^2)\Big(u_{xx} + \sinh(\sqrt{x^2 + y^2})u_{xxy}\Big) = 2021u_{xy} - 5$.

- ☐ Second order, nonlinear. It is quasilinear.
- ☐ Second order, linear.
- ☐ Third order, nonlinear. It is quasilinear.
- ☐ Third order, linear.

(c) In which *maximal* domain $\Omega \subset \mathbb{R}^2$ the second order linear PDE

$$x^2 u_{xx} + xyu_{xy} - yu_{yy} + \sin(x^2)u_x + 1 = 0,$$

is *elliptic*?

- ☐ $\Omega = \Big\{(x, y) \in \mathbb{R}^2 : x \neq 0 \text{ and } -1 < y < 0\Big\}$.
- ☐ $\Omega = \Big\{(x, y) \in \mathbb{R}^2 : x \neq 0 \text{ and } -4 < y < 0\Big\}$.
- ☐ $\Omega = \Big\{(x, y) \in \mathbb{R}^2 : x < -4 \text{ and } -1 < y < 0\Big\}$.
- ☐ $\Omega = \Big\{(x, y) \in \mathbb{R}^2 : 0 < x < 4 \text{ and } -4 < y < 0\Big\}$.

(d) Let $\alpha \in \mathbb{R}$ be an arbitrary constant. Then, the function

$$u(x, y) := \begin{cases} 0, & x < -\frac{\alpha^2}{3}y, \\ \alpha, & x > -\frac{\alpha^2}{3}y. \end{cases}$$

is a weak solution of the PDE

$$\begin{cases} u_y - u^2 u_x = 0, & \text{in } (x,y) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) = 0, & x < 0, \\ u(x,0) = \alpha, & x \geq 0. \end{cases}$$

For which values of α does u satisfy the *entropy condition*?

- ☐ $\alpha > 3$.
- ☐ $\alpha \neq 0$.
- ☐ $\alpha < 3$.
- ☐ $\alpha \in \mathbb{R}$ (i.e. for *any* α).

(e) Consider the Dirichlet problem

$$\begin{cases} \Delta u = 0, & \text{in } D, \\ u = \frac{x}{x^2+y^2} & \text{on } \partial D, \end{cases}$$

where the domain D is the anulus defined by $D := \left\{(x,y) \in \mathbb{R}^2 : 1 < \sqrt{x^2+y^2} < 2\right\}$. What is the maximum of u?

- ☐ $\frac{1}{2}$
- ☐ 1
- ☐ $\frac{1}{4}$
- ☐ -1

(f) Let $\alpha > 0$ be a fixed constant and $D := \{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} < \alpha\}$. For which value of $\beta \in \mathbb{R}$ is the Neumann problem

$$\begin{cases} \Delta u = \beta, & \text{in } D, \\ \partial_\nu u = 1, & \text{on } \partial D, \end{cases}$$

solvable?

- ☐ $\beta = \frac{2}{\alpha}$.
- ☐ $\beta = \frac{2}{\alpha^2}$.
- ☐ $\beta = \frac{\alpha}{2}$.
- ☐ $\beta = \frac{\pi}{\alpha}$.

2. Wave equation.

Let $u = u(x,t)$ be a solution of the wave equation

$$\begin{cases} u_{tt} - u_{xx} = 1, & (x,t) \in \mathbb{R}^2, \\ u(x,0) = f(x), & x \in \mathbb{R}, \\ u_t(x,0) = 0, & x \in \mathbb{R}, \end{cases}$$

where $f(x) = x^2$, if $|x| \leq 1$ and $f(x) = 0$, if $|x| > 1$.

(a) Where are the discontinuities of u and is $u(\cdot, t)$ odd/even/periodic? Justify your answer *before* solving point **(b)**.

(b) Compute the explicit solution u (distinguishing between cases if necessary), and compute $u(1, \frac{1}{2})$.

(c) Take advantage of the previous point to compute

$$\lim_{t \to +\infty} \frac{u(x,t)}{t^2},$$

for every $x \in \mathbb{R}$.

3. Method of Characteristics.

Consider the conservation law

$$\begin{cases} u_y + (u^3+1)u_x = 0, & x \in \mathbb{R},\ y > 0, \\ u(x,0) = h(x), & x \in \mathbb{R}, \end{cases}$$

where $h(x) = 1$ if $x \leq 0$, $h(x) = (1-x)^{\frac{1}{3}}$ if $0 < x < 1$ and $h(x) = 0$ if $x \geq 1$.

(a) Compute the critical time of existence y_c.

(b) Solve the above conservation law using the Method of Characteristics.

(c) Find a weak solution everywhere defined.

4. Separation of variables.

Solve the heat equation

$$\begin{cases} u_t - u_{xx} = \sin(x)\Big(\cos(t) + \sin(t)\Big) + \cos(x), & (x,t) \in (0,\pi) \times (0,+\infty), \\ u_x(0,t) = \sin(t), & t > 0, \\ u_x(\pi,t) = -\sin(t), & t > 0, \\ u(x,0) = 2\cos(x), & x \in [0,\pi]. \end{cases}$$

Hint: Take advantage of the function $w(x,t) := \sin(t)\sin(x)$ *to remove the non-homogeneous boundary conditions.*

5. Maximum principle

Let $Q := \Big\{(x,y) \in \mathbb{R}^2 : 0 < x < 1 \text{ and } 0 < y < 1\Big\}$, and let $u = u(x,y)$ be a function twice differentiable in Q and continuous in $\bar{Q}$, solving

$$\begin{cases} \Delta u(x,y) = -x^3 - y^3 - 6y^2, & \text{in } Q, \\ u(x,y) = g(x,y), & \text{on } \partial Q, \end{cases}$$

for some given function g. Show that

$$\max_{\bar{Q}} u \leq \frac{3}{5} + \max_{\partial Q} g.$$

Hint: Add a polynomial w to u to obtain an harmonic function.