4.1. Conservation laws and critical times Consider the PDE

$$u_y + \partial_x(f(u)) = 0\,.$$

In the following cases, compute the critical time y_c (i.e., the first time when the solution becomes nonsmooth):

(a) $f(u) = \frac{1}{2}u^2$, the initial datum is $u(x,0) = \sin(x)$.

The formula for the critical time is (if the internal infimum is negative)

$$y_c = -\Big(\inf_{x\in\mathbb{R}} \frac{d}{dx}\Big(f'(u(x,0))\Big)\Big)^{-1}\,.$$

Hence, we have

$$y_c = -\Big(\inf_{x\in\mathbb{R}} \cos(x)\Big)^{-1} = 1\,.$$

(b) $f(u) = \sin(u)$, the initial datum is $u(x,0) = x^2$.

The formula for the critical time is (if the internal infimum is negative)

$$y_c = -\Big(\inf_{x\in\mathbb{R}} \frac{d}{dx}\Big(f'(u(x,0))\Big)\Big)^{-1}\,.$$

Hence, we have

$$y_c = -\Big(\inf_{x\in\mathbb{R}} \frac{d}{dx}\cos(x^2)\Big)^{-1} = -\Big(\inf_{x\in\mathbb{R}} -2x\sin(x^2)\Big)^{-1} = 0\,,$$

thus the solution is singular for any positive time.

(c) $f(u) = e^u$, the initial datum is $u(x,0) = x^3$.

The formula for the critical time is (if the internal infimum is negative)

$$y_c = -\Big(\inf_{x\in\mathbb{R}} \frac{d}{dx}\Big(f'(u(x,0))\Big)\Big)^{-1}\,.$$

Hence, we have

$$y_c = -\Big(\inf_{x\in\mathbb{R}} \frac{d}{dx}e^{x^3}\Big)^{-1} = y_c = -\Big(\inf_{x\in\mathbb{R}} 3x^2e^{x^3}\Big)^{-1} = -(0)^{-1}\,,$$

since the internal infimum is nonnegative, we obtain that the solution remains smooth for all positive times (that is equivalent to saying that characteristic lines do not cross).

4.2. Weak solutions Consider the PDE

$$\partial_y u + \partial_x\left(\frac{u^4}{4}\right) = 0$$

in the region $x \in \mathbb{R}$ and $y > 0$.

(a) Show that the function $u(x,y) := \sqrt[3]{\frac{x}{y}}$ is a classical solution of the PDE.

Notice that $u_x = \frac{u}{3x}$ and $u_y = \frac{-u}{3y}$. Thus we have

$$u_y + \partial_x\left(\frac{u^4}{4}\right) = u_y + u_x u^3 = u_y + \frac{x}{y}u_x = \frac{-u}{3y} + \frac{x}{y}\frac{u}{3x} = 0\,.$$

(b) Show that the function

$$u(x,y) := \begin{cases} 0 & \text{if } x > 0, \\ \sqrt[3]{\frac{x}{y}} & \text{if } x \le 0. \end{cases}$$

is a *weak* solution of the PDE.

First of all, notice tha the function u is continuous.

Let us recall that a function u is a weak solution if for any $x_0 < x_1$ and any $0 < y_0 < y_1$, it holds

$$\int_{x_0}^{x_1} u(x,y_1) - u(x,y_0) + \int_{y_0}^{y_1} f(u(x_1,y)) - f(u(x_0,y)) = 0\,. \tag{1}$$

Since a classical solution is also a weak solution (check the last exercise of Serie 4 of the *old exercises* – you may find them on the website), thanks to what we have shown in part *(a)*, we already know that if $x_0 < x_1 \le 0$, then (1) holds. Since also the constant 0 is a classical solution of the PDE, we have that (1) holds also if $0 \le x_0 < x_1$.

It remains to prove the validity of (1) when $x_0 < 0 < x_1$. Thanks to what we have said above, we already know that (respectively setting $x_1 = 0$ and $x_0 = 0$)

$$\int_{x_0}^{0} u(x,y_1) - u(x,y_0) + \int_{y_0}^{y_1} f(u(0,y)) - f(u(x_0,y)) = 0\,,$$
$$\int_{0}^{x_1} u(x,y_1) - u(x,y_0) + \int_{y_0}^{y_1} f(u(x_1,y)) - f(u(0,y)) = 0\,.$$

Summing the two latter identities, we obtain exactly (1) for $x_0 < 0 < x_1$.