7.1. Nonhomogeneous wave equation Solve the following initial value problem:

$$\begin{cases} u_{tt} - u_{xx} &= 1, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) &= 1, & x \in \mathbb{R}, \\ u_t(x,0) &= 1, & x \in \mathbb{R}\,. \end{cases}$$

Let us apply d'Alembert's formula for the nonhomogeneous wave equation:

$$u(x,t) = \frac{1}{2}(1+1) + \frac{1}{2}\int_{x-t}^{x+t} 1\,ds + \frac{1}{2}\int_0^t \int_{x-t+\tau}^{x+t-\tau} 1\,ds\,d\tau = 1 + t + \frac{t^2}{2}\,.$$

7.2. Strange wave equation Show that the following partial differential equation admits a solution

$$\begin{cases} u_{tt} - u_{xx} &= \frac{u_t^2 - u_x^2}{2u}, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) &= x^4, & x \in \mathbb{R}, \\ u_t(x,0) &= 0, & x \in \mathbb{R}\,. \end{cases}$$

Hint: Consider the function $v(x,t) = \sqrt{u(x,t)}$. What equation does it satisfy?

We assume that u is a solution of the PDE and we compute the derivatives of $v(x,t) := \sqrt{u(x,t)}$. We have

$$\begin{aligned} v_t &= \frac{1}{2}u^{-\frac{1}{2}}u_t\,, \\ v_{tt} &= \frac{-1}{4}u^{-\frac{3}{2}}u_t^2 + \frac{1}{2}u^{-\frac{1}{2}}u_{tt}\,, \\ v_x &= \frac{1}{2}u^{-\frac{1}{2}}u_x\,, \\ v_{xx} &= \frac{-1}{4}u^{-\frac{3}{2}}u_x^2 + \frac{1}{2}u^{-\frac{1}{2}}u_{xx}\,. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} v_{tt} - v_{xx} &= \frac{-1}{4}u^{-\frac{3}{2}}u_t^2 + \frac{1}{2}u^{-\frac{1}{2}}u_{tt} + \frac{1}{4}u^{-\frac{3}{2}}u_x^2 - \frac{1}{2}u^{-\frac{1}{2}}u_{xx} \\ &= \frac{1}{2}u^{-\frac{1}{2}}(u_{tt} - u_{xx}) - \frac{1}{4}u^{-\frac{3}{2}}(u_t^2 - u_x^2) \\ &= \frac{1}{4}u^{-\frac{3}{2}}(u_t^2 - u_x^2) - \frac{1}{4}u^{-\frac{3}{2}}(u_t^2 - u_x^2) = 0\,. \end{aligned}$$

Hence we have proven that v satisfies

$$\begin{cases} v_{tt} - v_{xx} = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ v(x,0) = x^2, & x \in \mathbb{R}, \\ v_t(x,0) = 0, & x \in \mathbb{R}. \end{cases} \tag{1}$$

Up to now, we have just noticed that if u is a solution of the original PDE then $\sqrt{u}$ solves the wave equation; but we have never shown the existence of u. In order to do so, let $v(x,t)$ be the solution of (1) and define $u := v^2$ (our computations justify this choice). We want to check that u is a solution of the original PDE.

Thanks to the d'Alembert's formula we know

$$v(x,t) = x^2 + t^2$$

and therefore

$$u(x,t) = x^4 + t^4 + 2x^2t^2$$

is our candidate solution. Checking the initial conditions $u(x,0) = x^4$ and $u_t(x,0) = 0$ is immediate. Hence, we just have to check whether u solves the equation. We have

$$\begin{aligned} u_{tt} - u_{xx} &= 12t^2 + 4x^2 - (12x^2 + 4t^2) = 8(t^2 - x^2)\,, \\ u_t &= 4t^3 + 4x^2t = 4t(x^2 + t^2) \implies u_t^2 = 16t^2u\,, \\ u_x &= 4x^3 + 4tx^2 = 4x(x^2 + t^2) \implies u_x^2 = 16x^2u\,. \end{aligned}$$

Therefore we get

$$u_{tt} - u_{xx} = 8(t^2 - x^2) = \frac{16t^2u - 16x^2u}{2u} = \frac{u_t^2 - u_x^2}{2u}$$

which is exactly the desired partial differential equation. Thus we have shown that $u(x,t) = x^4 + t^4 + 2x^2t^2$ is a solution.

7.3. Symmetries Let $u : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ be a solution of the wave equation

$$\begin{cases} u_{tt} - c^2u_{xx} = F(x,t), & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = f(x), & x \in \mathbb{R}, \\ u_t(x,0) = g(x), & x \in \mathbb{R}. \end{cases} \tag{2}$$

By means of the uniqueness of the solution of the wave equation, show that

(a) If f, g, and F are spatially odd (odd with respect to x), then u is also spatially odd. (That is, $f(-x) = -f(x)$, $g(-x) = -g(x)$ and $F(-x,t) = -F(x,t)$ imply $u(-x,t) = -u(x,t)$.)

Hint: Consider the function $v(x,t) = -u(-x,t)$. What equation does it satisfy?

We proceed with the hint. Take $v(x,t)$. Notice that $v_t(x,t) = -u_t(-x,t)$, $v_{tt}(x,t) = -u_{tt}(-x,t)$ on the one hand, and $v_x(x,t) = u_x(-x,t)$, and $v_{xx}(x,t) = -u_{xx}(-x,t)$. Thus,

$$v_{tt}(x,t) - c^2 v_{xx}(x,t) = -u_{tt}(-x,t) + c^2 u_{xx}(-x,t) = -F(-x,t) = F(x,t),$$

where in the last equality we are using that F is spatially odd. Similarly,

$$v(x,0) = -u(-x,0) = -f(-x) = f(x), \quad v_t(x,0) = -u_t(-x,0) = -g(-x) = g(x).$$

Thus, v satisfies

$$\begin{cases} v_{tt} - c^2 v_{xx} &= F(x,t), & (x,t) \in \mathbb{R} \times (0,\infty), \\ v(x,0) &= f(x), & x \in \mathbb{R}, \\ v_t(x,0) &= g(x), & x \in \mathbb{R}, \end{cases}$$

that is, v and u satisfy the same problem, (2). Since (2) has a unique solution, we must have $v(x,t) = u(x,t)$ for all $x \in \mathbb{R}$, $t \geq 0$; that is, $-u(-x,t) = u(x,t)$, u is spatially odd.

(b) If f, g, and F are spatially even (even with respect to x), then u is also spatially even. (That is, $f(-x) = f(x)$, $g(-x) = g(x)$ and $F(-x,t) = F(x,t)$ imply $u(-x,t) = u(x,t)$.)

The solution is the same as above, taking $v(x,t) = u(-x,t)$ instead.

(c) If f, g, and F are L-periodic, then u is also L-periodic. (That is, $f(x) = f(x+L)$, $g(x) = g(x+L)$ and $F(x,t) = F(x+L,t)$, imply $u(x,t) = u(x+L,t)$.)

The solution is the same as above, taking $v(x,t) = u(x+L,t)$ instead.

7.4. Wave equation on a ring Let $u : [0,1] \times [0,\infty) \to \mathbb{R}$ be a solution of the wave equation

$$\begin{cases} u_{tt} - u_{xx} &= 0, & (x,t) \in [0,1] \times (0,\infty), \\ u(x,0) &= x - x^2, & x \in [0,1], \\ u_t(x,0) &= 0, & x \in [0,1], \\ u(0,t) &= u(1,t), & t \in (0,\infty), \\ u_x(0,t) &= u_x(1,t), & t \in (0,\infty). \end{cases}$$

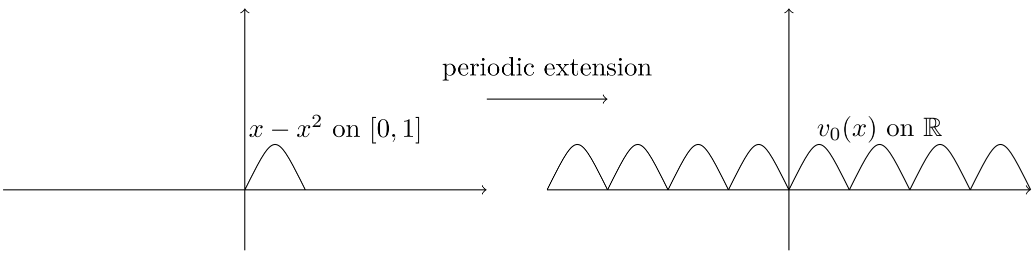
Compute $u(\frac{1}{2}, 2021)$.

A general consideration: Up to now, the only tool we have to explicitly compute u is the d'Alembert formula, which can be applied only when the PDE takes place on the whole real line $\mathbb{R}$. In general one can be interested in solving a wave equation taking place in a smaller domain $D \subset \mathbb{R}$. The trick is the following: we *extend* the boundary data of the PDE on the whole line producing an auxiliary problem that we can solve with d'Alembert. Then, we restrict the computed solution on D, and we check taking advantage of Exercise 7.3 that the restriction solves the original PDE. The way the extension should be operated is suggested by the additional boundary conditions of the problem. For instance, exercise 6.3 was solved in this way. This exercise is another example of this general method.

In this particular case we want to solve the wave equation on $[0,1]$, and we search for a PDE on the whole line of the form

$$\begin{cases} v_{tt} - v_{xx} &= 0, \qquad (x,t) \in \mathbb{R} \times (0,\infty), \\ v(x,0) &= v_0(x), \qquad x \in \mathbb{R}, \\ v_t(x,0) &= 0, \qquad x \in \mathbb{R}\,, \end{cases} \tag{3}$$

such that setting $u(x,t) := v(x,t)$ for $(x,t) \in [0,1] \times (0,\infty)$, u solves the original problem. The question now is how v_0 should be defined. Of course $v_0(x,t) = x - x^2$ for $x \in [0,1]$ since we want that $v\mid_{[0,1]} = u$. The additional boundary conditions $u(0,t) = u(1,t)$ and $u_x(0,t) = u_x(1,t)$ are imposing the solution to be *periodic*. The natural way to define v_0 is therefore by periodicity: $v_0(x) = (x - \lfloor x \rfloor) - (x - \lfloor x \rfloor)^2$, where $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : 0 \le x - n < 1\}$. This is just a fancy notation for the natural periodic extension showed in the picture below



We know that the solution exists and is periodic (with period 1) as shown in *(c)* of the previous exercise. Let u be the restriction of v in the domain $[0,1] \times [0,\infty)$. Clearly u satisfies the wave equation in the domain and $u(x,0) = v(x,0) = x - x^2$ and $u_t(x,0) = v_t(x,0) = 0$. Moreover, thanks to the periodicity of v, we have

$$\begin{aligned} u(0,t) = v(0,t) = v(1,t) = u(1,t)\,, \\ u_x(0,t) = v_x(0,t) = v_x(1,t) = u_x(1,t)\,. \end{aligned}$$

Thus the function u satisfies the PDE given in the statement. With a similar argument one can also prove that this u is the unique solution (the idea is to define v as the extension of u and show that it satisfies (3), so that we can invoke the uniqueness for the classical wave equation).

To compute the value of $u(\frac{1}{2}, 2021)$ we exploit the d'Alembert formula for v:

$$u(\tfrac{1}{2}, 2021) = v(\tfrac{1}{2}, 2021) = \frac{1}{2}(v_0(\tfrac{1}{2} - 2021) + v_0(\tfrac{1}{2} + 2021)) = v_0(\tfrac{1}{2}) = \frac{1}{4}\,.$$

7.5. Multiple Choice Determine the correct answer.

(a) Consider the modified one dimensional wave equation of Problem 6.5 with nonhomogeneous right hand side

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(t), & \\ u(x,0) = \arctan(x), & x \in \mathbb{R}, \\ u_t(x,0) = 0, & x \in \mathbb{R}. \end{cases}$$

Suppose also that there exists a function $m = m(t) \in C^2$ such that $m_{tt} = F$ and $m(0) = m_t(0) = 0$. Then, without applying directly the d'Alembert formula (but possibly using Problem 6.5), the asymptotic value of u as $t \to \infty$, i.e. $\lim_{t\to+\infty} u(\bar{x}, t)$, is well defined and finite for all $\bar{x} \in \mathbb{R}$

- ☐ always because it is equal to 0.
- ☐ if $\lim_{t\to+\infty} m(t)$ exists and it is finite.
- ☐ if $\lim_{t\to+\infty} m'(t)$ exists and it is finite.

Let w be the solution of the homogeneous problem

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0, & \\ w(x,0) = \arctan(x), & x \in \mathbb{R}, \\ w_t(x,0) = 0, & x \in \mathbb{R}. \end{cases}$$

Then, it is immediate (by direct computation or superposition principle) to see that $u := w + m$ solves the nonhomogeneous problem with F as right hand side. We already know from Problem 6.5 that w has asymptotic value equal to zero. Therefore, the correct solution is the second one since

$$\lim_{t\to+\infty} u(\bar{x}, t) = \lim_{t\to+\infty} \Big(w(\bar{x}, t) + m(t)\Big) = \lim_{t\to+\infty} m(t).$$