Analysis III, 2019-2020,
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Extra 2

1 Revision of ODEs

Since the method of characteristics reduces PDEs to ODEs, we start with a quick review of the ODEs that will be relevant for us.

1.1 Methods for solving first-order scalar ODEs

In this section we recall how to solve separable ODEs, which have the form

$$\frac{dx}{dt} = f(x)g(t), \tag{1}$$

and first-order linear ODEs, which have the form

$$\frac{dx}{dt} + f(t)x = g(t). \tag{2}$$

Remark: It is not worth memorising the formulas. It is easier to simply derive them whenever needed.

1.2 1st order, separable

Let $I \subseteq \mathbb{R}$ be an open interval containing 0 and let $x \in C^1(I)$ satisfy the separable ODE

$$\begin{aligned} &\dot{x}(t) = f(x(t))g(t), \quad t \in I, \\ &x(0) = x_0, \end{aligned}$$

where $f, g \in C^1(\mathbb{R})$. Let h be a primitive of $1/f$, i.e., let h satisfy $\dot{h}(t) = 1/f(t)$. Assume that $f(x(t)) \neq 0$ for all $t \in I$.

Since $f(x(t)) \neq 0$, we can divide the ODE by $f(x(t))$ to obtain

$$\begin{aligned} \dot{x}(t) = f(x(t))g(t) \quad &\iff \quad \frac{\dot{x}(t)}{f(x(t))} = g(t) \\ &\iff \quad \int_0^t \frac{\dot{x}(s)}{f(x(s))}\, ds = \int_0^t g(s)\, ds \\ &\iff \quad \int_{x(0)}^{x(t)} \frac{1}{f(X)}\, dX = \int_0^t g(s)\, ds \quad \text{(change variables: } X = x(s),\, dX = \dot{x}(s)\, ds) \\ &\iff \quad \int_{x(0)}^{x(t)} \dot{h}(X)\, dX = \int_0^t g(s)\, ds \\ &\iff \quad h(x(t)) - h(x(0)) = \int_0^t g(s)\, ds \\ &\iff \quad h(x(t)) = h(x_0) + \int_0^t g(s)\, ds. \end{aligned}$$

Observe that $h'(x(t)) = 1/f(x(t)) > 0$ by assumption. Therefore h is invertible in a neighbourhood of $x(t)$, and so the formula for the solution is given by

$$x(t) = h^{-1}\left(h(x_0) + \int_0^t g(s)\,ds\right).$$

1.3 1st order, linear

Let $x \in C^1(\mathbb{R})$ satisfy the first-order linear ODE

$$\begin{aligned} \dot{x}(t) + a(t)x(t) &= b(t), \quad t \in \mathbb{R}, \\ x(0) &= x_0, \end{aligned}$$

where $a, b \in C^1(\mathbb{R})$. Let A be a primitive of a, i.e., let A satisfy $\dot{A}(t) = a(t)$. This is sometimes denoted as $A(t) = \int a(t)\,dt$.

Multiplying the ODE by the integrating factor $e^{A(t)}$ gives

$$\begin{aligned} e^A\dot{x} + e^A a x = e^A b \quad &\iff \quad \frac{d}{dt}\left(e^A x\right) = e^A b \\ &\iff \quad \int_0^t \frac{d}{ds}\left(e^A x\right)\,ds = \int_0^t e^{A(s)}b(s)\,ds \\ &\iff \quad e^{A(t)}x(t) - e^{A(0)}x(0) = \int_0^t e^{A(s)}b(s)\,ds \\ &\iff \quad x(t) = e^{A(0)-A(t)}x_0 + e^{-A(t)}\int_0^t e^{A(s)}b(s)\,ds. \end{aligned}$$

This provides the general formula for the solution.

Example 1 (Using an integrating factor) *Solve the ODE*

$$\begin{aligned} \dot{x} &= \lambda x, \\ x(0) &= x_0. \end{aligned}$$

Here $\dot{x}$ denotes the derivative dx/dt. We'll treat this as a first-order linear ODE rather than as a separable ODE, but either method works. This equation has the form of (2) with $f(t) = -\lambda$, $g(t) = 0$. Multiplying the ODE by the integrating factor

$$\exp\left(\int f(t)\,dt\right) = \exp(-\lambda t)$$

gives

$$e^{-\lambda t}\dot{x} - \lambda e^{-\lambda t}x = 0 \quad \iff \quad \frac{d}{dt}\left(e^{-\lambda t}x\right) = 0.$$

Now we just integrate and use the Fundamental Theorem of Calculus:

$$0 = \int_0^t \frac{d}{dt}\left(e^{-\lambda t}x(t)\right)\,dt = e^{-\lambda t}x(t) - e^{-\lambda \cdot 0}x(0) \quad \iff \quad x(t) = x_0 e^{\lambda t}.$$

We will often encounter ODEs of the form $\dot{x} = \lambda x$ and it is worth memorising the solution $x(t) = x(0)e^{\lambda t}$; you should not need to derive it every time.

Example 2 (Existence for only a finite time) *Solve the ODE*

$$\dot{x} = x^2,$$
$$x(0) = 1.$$

Using the method of separation gives

$$\begin{aligned}
\dot{x} = x^2 \quad &\iff \quad \frac{1}{x^2}\,\dot{x} = 1 \\
&\iff \quad \int_0^t \frac{1}{x^2(s)}\,\dot{x}(s)\,ds = \int_0^t 1\,ds \\
&\iff \quad \int_{x(0)}^{x(t)} \frac{1}{X^2}\,dX = t \qquad\qquad (X = x(s),\ dX = \dot{x}(s)\,ds) \\
&\iff \quad -\frac{1}{X}\bigg|_{x(0)}^{x(t)} = t \\
&\iff \quad -\frac{1}{x(t)} + 1 = t \\
&\iff \quad x(t) = \frac{1}{1-t}.
\end{aligned}$$

The solution blows up at time $t = 1$: $x(t) \to \infty$ as $t \to 1$.

Even though the function $x(t) = (1-t)^{-1}$ is defined for all $t \neq 1$, we say that ODE only has a solution up until (and not including) time 1.

1.4 Linear second-order ODEs

Consider the second-order, linear, constant-coefficient ODE

$$\frac{d^2x}{dt^2} + c^2 x = 0.$$

This has solutions of the form

$$x(t) = A\cos(ct) + B\sin(ct),$$

where A, B are constants. On the other hand, the general solution to the ODE

$$\frac{d^2x}{dt^2} - c^2 x = 0$$

is given by

$$x(t) = A\sinh(ct) + B\cosh(-ct),$$

where

$$\sinh(t) = \frac{e^t - e^{-t}}{2}, \qquad \cosh(t) = \frac{e^t + e^{-t}}{2}.$$

Note that $\sinh'(t) = \cosh(t)$ and $\cosh'(t) = \sinh(t)$.
Remark: the solution to the ODE

$$\frac{d^2x}{dt^2} - c^2 x = 0$$

can also be written as

$$x(t) = C\exp(ct) + D\exp(-ct).$$

However, as we will see later, sinh and cosh are more convenient when imposing boundary conditions, since $\sinh(0) = 0$ and $\cosh'(0) = 0$.

2 An example of solution: The method of characteristics for a linear PDE.

Solve the linear first-order PDE

$$\begin{aligned} u_x + u_y + u &= 1 \quad \text{for } (x,y) \in \mathbb{R}^2, \\ u &= x^2 \quad \text{for } (x,y) \in \mathbb{R} \times \{0\}. \end{aligned}$$

The PDE has the form

$$a_1(x,y,u(x,y))u_x(x,y) + a_2(x,y,u(x,y))u_y(x,y) = b(x,y,u(x,y))$$

with

$$a_1(x,y,z) = 1, \quad a_2(x,y,z) = 1, \quad b(x,y,z) = 1 - z.$$

The value of u is prescribed on the line $\mathbb{R} \times \{0\}$, which we can parametrise by $\gamma(s) = (x_0(s), y_0(s)) = (s, 0)$, $s \in \mathbb{R}$. Define

$$u_0(s) = x_0^2(s) = s^2.$$

Hence the initial condition is given by

$$\Gamma = \{(s, 0, s^2) \, : \, s \in \mathbb{R}\}.$$

Step 1: We need to solve

$$\frac{d}{dt}x = a(x,y,\tilde{u}) = 1, \tag{3}$$

$$\frac{d}{dt}y = b(x,y,\tilde{u}) = 1, \tag{4}$$

$$\frac{d}{dt}\tilde{u} = c(x,y,\tilde{u}) = 1 - \tilde{u}, \tag{5}$$

subject to the initial conditions

$$x(0,s) = x_0(s) = s, \tag{6}$$

$$y(0,s) = y_0(s) = 0, \tag{7}$$

$$\tilde{u}(0,s) = u_0(s) = s^2. \tag{8}$$

Equations (3), (6) imply that

$$x(t,s) = t + s.$$

Equations (4), (7) imply that

$$y(t,s) = t.$$

Multiply equation (5) by the integrating factor

$$\exp\left\{\int 1 \, dt\right\} = e^t$$

to obtain

$$e^t \tilde{u}_t + e^t \tilde{u} = e^t \quad \Longleftrightarrow \quad \frac{d}{dt}(e^t \tilde{u}) = e^t.$$

Integrating from 0 to t gives

$$e^t\tilde{u}(t,s) - e^0\tilde{u}(0,s) = \int_0^t e^\tau \, d\tau \quad \Longleftrightarrow \quad e^t\tilde{u}(t,s) - s^2 = e^t - 1 \quad \Longleftrightarrow \quad \tilde{u}(t,s) = 1 + e^{-t}(s^2 - 1).$$

Step 2: We need to invert the map $(t,s) \mapsto (x(t,s), y(t,s)) = (t+s, t)$. Setting $(x,y) = (t+s,t)$ and solving for t and s in terms of x and y gives $t = y$, $s = x - t = x - y$. Therefore

$$t(x,y) = y, \quad s(x,y) = x - y.$$

Step 3: Finally,

$$u(x,y) = \tilde{u}(t(x,y), s(x,y)) = \boxed{1 + e^{-y}((x-y)^2 - 1)}$$

It is easy to check that u satisfies the Cauchy problem.

Plotting the characteristics: The (projection of the) characteristics are the curves

$$t \mapsto (x(t,s), y(t,s)) = (t+s, t) = (s,0) + t(1,1),$$

which are in fact straight lines. We can write these lines in nonparametric form as $y = x - s$. Some representative characteristics are plotted below.