Analysis III

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Main objective: learn how to solve some of the most important types op PDEs
partial differential equations

Reference: Pinchover - Rubinstein, "An introduction to partial differential equations"

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Präsenz:

Exercise classes:

(⚠ the numbers of equations start again every new lecture)

- Why PDEs?

The theory of PDEs has been developed in the last three centuries to solve the most fundamental problems in physics, engineering and other sciences.
From the study of the propagation of waves in a fluid, to the flow of heat or to the electrical networks.
Indeed most of the fundamental laws of the physical sciences are PDEs.
The study of PDEs is a huge field and with the variety of possible PDEs it is impossible to find a method by which we can solve all equations

This is actually a very active area of modern mathematics and even if there have been dramatic progress in the last 50 years, there are still many open problems and there exist very complex equations that cannot yet be solved even with the aid of supercomputers.

In this course we will touch upon basic (yet fundamental!) techniques for certain important classes of PDEs, but we will only skim the surface of this field.

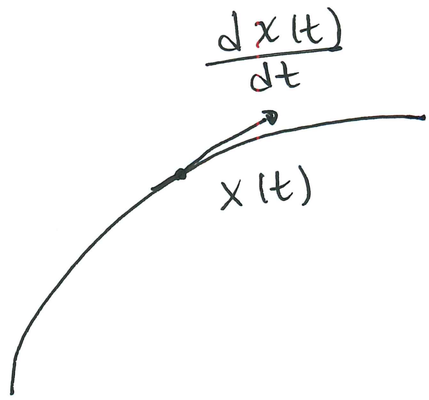
As you may already know, a differential equation is an equation for some unknown function involving one or more derivatives of the unknown function:

ODE = ordinary differential equation = equation involving functions of ONE independent variable and one or more of their derivatives.

Example $\quad m \dfrac{d^2 x(t)}{dt^2} = F(x(t))$ Newton's second law

Unknown function: $x(t) = (x_1(t), x_2(t), x_3(t))$ represents the position of a particle at time t

- $\dfrac{dx(t)}{dt}$ = velocity vector
- $\dfrac{d^2 x(t)}{dt^2}$ = acceleration
- $m \in \mathbb{R}^+$ = mass
- $F(x)$ = force field



Preliminaries (Section 1.1 PR)

PDE = partial differential equation = equation involving an unknown function u of MORE than ONE variable and certain of its partial derivatives.

Hereafter u will denote the real-valued solution (i.e. unknown) of a given PDE and it is usually a function of points $x = (x_1, x_2, .., x_n) \in \mathbb{R}^n$ typically denoting a position in space, and sometimes also a function of $t \in \mathbb{R}$, denoting time.

We will also use x, y, z to denote independent variables (instead of $x_1, x_2, x_3 ...$)

Notation: we write $u_{x_k} = \dfrac{\partial u}{\partial x_k}$ to denote the partial derivative of u with respect to x_k,

$$u_t = \frac{\partial u}{\partial t}, \quad u_{x_k x_e} = \frac{\partial^2 u}{\partial x_k \partial x_e} \quad \dots \qquad \left(\text{If } u = u(x,y) \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y} \dots \right)$$

We recall the following theorem:

THEOREM (Schwartz) Given a function u continuously differentiable often enough, then the order of partial derivatives is irrelevant, for instance if $u = u(x,y)$, $u_{xy} = u_{yx}$.

If $u = u(x,y,z)$, the gradient of u is:

$$\nabla u = (u_x, u_y, u_z)$$

and the Laplacian of u is:

$$\Delta u = u_{xx} + u_{yy} + u_{zz}$$

We will now see some example that will also motivate the use of partial differential equations: when we want to study a physical system we want to understand the state of such system at any point in space and at any time.

Example suppose that $u(x,y,z,t)$ is the temperature at the point (x,y,z) and at time t. We know that this state changes over time, so we will consider the quantity u_t = change w.r.t. t

But it will also change w.r.t. the position so we will consider partial derivatives of u w.r.t. x, y and z : u_x, u_y, u_z, u_{xx}, u_{xy} ...

Surely we need to relate the variations in space and in time and it turns out that the heat flow over time may be described by:

$$u_t = k(u_{xx} + u_{yy} + u_{zz}) = k\,\Delta u$$

$u_t = \Delta u$ Heat equation

Example: Another fundamental equation that we will see later in the course is the Laplace equation that records diffusion effects in equilibrium:

$\Delta u = 0$ Laplace's equation

Example : the wave equation

$$u_{tt} = c^2 \Delta u$$

superficially resembles the heat equation but it supports solutions with a completely different behaviour and can be used to describe the propagation of a wave in a fluid.

Example : Burgers equation

$$u_t = u u_x$$

can model the flow of a viscous fluid or the traffic flow and it is a prototypical example of conservation law.

• What is a well-posed problem? (Sections 1.1 ~~→~~ 1.5)

In general we study equations that originate from a physical or engineering problem:

Real life problem ⇝ model ⇝ PDE

It is not obvious that that the model is consistent in the sense that it leads to a solvable PDE.

Furthermore we wish the solution to be unique and to be stable under small perturbations of the data:

By a "problem" we mean a PDE supplemented initial or boundary conditions. A problem is well posed if it satisfies the following criteria:

1. the problem has a solution (EXISTENCE)
2. the solution is unique (UNIQUENESS)
3. A small change in the equation and/or in the side conditions gives rise to a small change in the solution (STABILITY)

If one or more of those conditions does not hold, then the problem is said to be ill-posed.

Initial and boundary conditions: As you may recall from studying ODEs, there may be no solutions or there may be many solutions for a given ODE(*). The same is true for PDEs. Indeed PDEs have in general infinitely many solutions and in order to obtain a unique solution we must supplement the equation with additional conditions.

What kind of conditions? This depends by the type of PDE under study.

Example (Transport equation) Consider the transport equation $u_t + c u_x = 0$, $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $c \in \mathbb{R}$.

If $u \geq 0$, u can represent the concentation of a pollutant in a river at time t and position x. $c \in \mathbb{R}$ represents the velocity of the river and in order to have a complete information about u in time it makes sense to couple this equation with an information about the concentration of the pollutant at time zero:

$$(\text{IVP}) \begin{cases} u_t + c u_x = 0, & (x,t) \in \mathbb{R} \times (0, +\infty) \\ u(x,0) = g(x), & x \in \mathbb{R} \end{cases}$$

Initial value problem

$g > 0$ is the concentation of the pollotant at time zero.

(*) if the initial conditions are not appropriate

Example : (wave eq. in 1D, vibrating string)

In the case of the 1D- vibrating string we will have the wave equation with four side conditions :

$$\begin{cases} u_{tt} = u_{xx} \quad , x \in (0,L) , \; t \in \mathbb{R}^+ \\ u(0,t) = u(L,t) = 0 \quad , t \geq 0 \\ u(x,0) = f(x) \quad , x \in [0,L] \\ u_t(x,0) = g(x) \quad , x \in [0,L] \end{cases}$$

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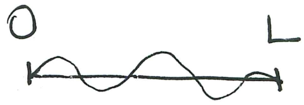

the second equation expresses two boundary conditions (the string is fixed at positions 0 and L) and the last two equations express the initial conditions : they tell us what happens at time zero in terms of deflection $f(x)$ and of speed $g(x)$.

Remark : the domain of the PDE is defined only in the interior of the domain because the function u may not be differentiable on the boundary.

Definition : (Strong solution) The solution of a PDE is called strong if all the derivatives of the solution that appear in the PDE exist and are continuous. Otherwise the solution is called weak.

Weak solutions have points in their domain where the derivatives do not exist (or are not continuous), so a weak solution cannot directly be plugged into the equation.

Remark : there is no universal definition of weak solution, but the definition depends by the type of PDE and we will see this later when studying conservation laws.

Classification and properties of PDEs

(Sections 1.2, 1.3)

Definition (ORDER of a PDE) the order of a PDE is the order of the highest order partial derivative of the unknown appearing within it.

Example:

- $u_x + u^2 u_{yy} = e^x u_{xy}$ has order 2
- $u_{xyz} = xy^2 + zu$ has order 3
- $u_{tt} = u_{xx} + f(x,t)$ has order 2

Note: we will mostly work with PDEs of first and second order.

[Recall linear operator $u \rightsquigarrow \boxed{\mathcal{L}} \rightsquigarrow \mathcal{L}[u]$: $\mathcal{L}[\alpha u_1 + \beta u_2] = \alpha \mathcal{L}[u_1] + \beta \mathcal{L}[u_2]$.]

Definition (LINEARITY): A PDE is called linear if it is of the form

$$(1) \quad \overbrace{a^{(0)} u + \sum_{i=1}^{n} a_i^{(1)} u_{x_i} + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^{(2)} u_{x_i x_j} + \dots}^{\mathcal{L}[u]} = f(x)$$

where $f(x)$ and $a_{ij}^{(m)}$ are functions of the variable $x = (x_1, \dots, x_n)$.

A PDE is linear if every summand consists of a function multiplied by u or by one of its derivatives.

Equivalently, if u and v solve (1), then $u - v$ solve (1) with $f = 0$.

Examples:

.) $xy u_x + \sin^2(y) u_{xy} - e^x u_{yy} = 2xy^3$ LINEAR

..) $\underbrace{u u_x} = 2$ NON LINEAR

...) $u_t = u_x + \underbrace{u^2}$ NON LINEAR

IV) $u_{tt} = u_{xxxx}$ LINEAR

Definition (Homogeneity) A linear PDE of the form as defined in (1) is called homogeneous if $f(x) = 0$. When $f \neq 0$ we say that the PDE is inhomogeneous and the function $f(x)$ is called inhomogeneity

Important property: If we have a LINEAR HOMOGENEOUS PDE with two solutions u_1 and u_2, then any linear combination of u_1 and u_2 is a solution as well.
(u_1, u_2 solutions $\Rightarrow \alpha u_1 + \beta u_2$, $\alpha, \beta \in \mathbb{R}$ solution)

Theorem (Vector space of solutions) Let $\mathcal{L}[u] = f(x)$ be a linear inhomogeneous PDE and $\mathcal{L}[u] = 0$ the corresponding homogeneous PDE.

Let u_1, u_2 be solutions of $\mathcal{L}[u] = 0$ and u_p be a solution of $\mathcal{L}[u] = f(x)$.
Then for all $\alpha, \beta \in \mathbb{R}$ we have that $\alpha u_1 + \beta u_2$ is a solution of $\mathcal{L}[u] = 0$ and $\alpha u_1 + \beta u_2 + u_p$ is a solution of $\mathcal{L}[u] = f(x)$.

Remark: we can denote $\mathcal{L}[u]$ a linear operator acting on u, for example $\mathcal{L}[u] = \mathcal{L}[x, t, p, q] = q - p$ so that the transport equation can be written as: $\mathcal{L}[x, t, u_x, u_t] = 0$:

$$\mathcal{L}[x, t, u_x, u_t] = u_t - u_x = 0$$

Example: .) $u_t + u_x = 0$ LINEAR, HOMOGENEOUS

..) $u_{xx} + u_{yy} = x^2 + y^2$ LINEAR, INHOMOGENEOUS

...) $u_x^2 + u_y^2 = 1$ NON LINEAR

As you may be able to guess, many PDEs are not linear and non linear equations are often further classified into subclasses according to the type of nonlinearity.

We will sometimes handle non linear PDEs which still have a special structure called quasi-linearity.
The main reason to make all these distinctions lies in the tools available to solve such problems.
For example, since next Monday, we will start studying the method of characteristics that will allow us to solve first order quasilinear PDEs.

Definition (QUASI-LINEARITY) A PDE is called quasi-linear if it is linear in its highest order derivative term.

Example:

1. $u u_x + x^2 u_y = u^3$ quasi-linear, first order

2. $u_x + u u_y = 0$ quasi-linear, first order

3. $u u_x + u^2 u_y + u = e^u$ quasi-linear, first order

4. $\underbrace{u_x u_y} = x^2 y$ not quasi-linear, first order

   two first order derivatives are multiplied

5. $u_t = u_y u_{xx} + u^2 u_{yy} + u_x^2$ quasi linear, second order

6. $\underbrace{u_{xy}^2} = xu + u_y$ not quasi linear, second order

   the second order derivative u_{xy} does not appear in a linear manner.

We will conclude with a modelling example

Example (From stock markets to the heat eq.)

Let us model a stock market.

$$\begin{cases} Y(t) = \text{price of an asset} \\ Y(0) = 1 \end{cases}$$
IC (we assume an initial value w.l.o.g.)

Asset prices grow and decay exponentially so we look at:

$$\begin{cases} X(t) := \log(Y(t)) - rt, \quad \text{where } r>0 \text{ is the growth rate.} \\ X(0) = 0 \end{cases}$$
$\rightsquigarrow$ at time 0 the value of X is given by 0 with probability 1.

We now analyse the evolution of $X(t)$ using the Merton model:

We assume that, given a time step $\tau>0$, the value at time $t+\tau$ is given by:

$$X(t+\tau) = X(t) \pm \delta, \quad \delta>0$$

and one adds or subtracts δ with probability $\frac{1}{2}$ respectively. Let us denote $\text{Prob}(X(t)=x) = p(x,t)$. Then the equation for $X(t+\tau)$ gives

$$p(x,t+\tau) = \frac{1}{2}p(x+\delta,t) + \frac{1}{2}p(x-\delta,t)$$

therefore rearranging the terms

$$\frac{p(x,t+\tau) - p(x,t)}{\tau} = \frac{\delta^2}{2\tau}\,\frac{p(x+\delta,t) + p(x-\delta,t) - 2p(x,t)}{\delta^2}$$

Assuming that $\frac{\delta^2}{2\tau} \longrightarrow K>0$, by taking the limit $\tau\to 0$ and $\delta\to 0$ we get

$$p_t(x,t) = K\,p_{xx}(x,t)$$

So the probability density must fulfill the heat eq.

The initial condition is given by $p(x,0) = \delta(0)$

The solution to this equation is well-known and it is given by:

$$p(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-\frac{x^2}{4\pi k t}}$$

Summary:

- PDEs are normally used together with boundary conditions
- Well-posedness stands for the existence and uniqueness of a stable solution
- Linearity stands for linear in every appearence of u, while quasilinearity stands for linear in the highest order derivatives of u
- A linear PDE is homogeneous if all the terms depend linearly on u or on its derivatives (in other words there is no right hand side)