Recall of lecture 1 :

Def : A partial differential equation is an equation involving an unknown function u of more than one independent variable and certain of its partial derivatives

Example : $u_t = \Delta u$ (Heat equation)

$$u = u(x_1, x_2, x_3, t)$$

$$\Delta u = u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3}$$

* We usually supplement a PDE with initial or boundary condition, and we call this a Cauchy problem. A Cauchy problem is well-posed if
  1. the problem has a solution
  2. the solution is unique
  3. the solution is stable

* Classification of PDEs :
  - ORDER = Order of the highest order derivative appearing in the PDE : $u_t = \underbrace{u_{xxx}}_{\text{3rd order}}$
  - LINEAR PDE : the unknown appears linearly

    $u_t = u_{xx} + u_{xy}$ 2nd order linear

  - QUASILINEAR PDE : highest order terms appear linearly

$$\underbrace{(u_t)^2}_{\text{non linear but 1st order}} + \underbrace{u_{xxx}}_{\text{3rd order term, appears linearly}} + \overbrace{u_x u_y}^{\text{non linear term, but first order.}} = 0$$

Lecture 2 30/09/2019

- In this lecture we will introduce an approach to solve PDEs known as method of characteristics (MoC)
  which kind?
- This method will be used to solve first order quasilinear equations ⤳ this is why the classification is important
- The MoC relies on a powerful geometrical interpretation of first order PDEs ⤳ more evident for PDEs of 2 variables
- The MoC reduces a scalar PDE to a system of ODEs.

First-order equations (Sections 2.1)

A first-order PDE for an unknown function $u(x_1, \dots, x_n)$ can be written in general form

$$(1) \quad F(\underbrace{x_1, x_2, \dots, x_n}_{n}, \underbrace{u}_{1}, \underbrace{u_{x_1}, \dots, u_{x_n}}_{n}) = 0$$

where F is a given function of $2n+1$ variables.

* Evolution of a pollutant concentration in a channel, or traffic dynamics, optics.

We will consider two-dymensional real-valued functions $u(x,y)$ for which (1) reduces to

$$(2) \quad F(x, y, u, u_x, u_y) = 0$$

These equations establish a relation between the solution surface to its tangent plane. Indeed since $u(x,y)$ is a SURFACE in $\mathbb{R}^3$, and since the normal to this surface is parallel to the vector $(u_x, u_y, -1)$, then (2) relates the equation to its normal (and tangent plane).

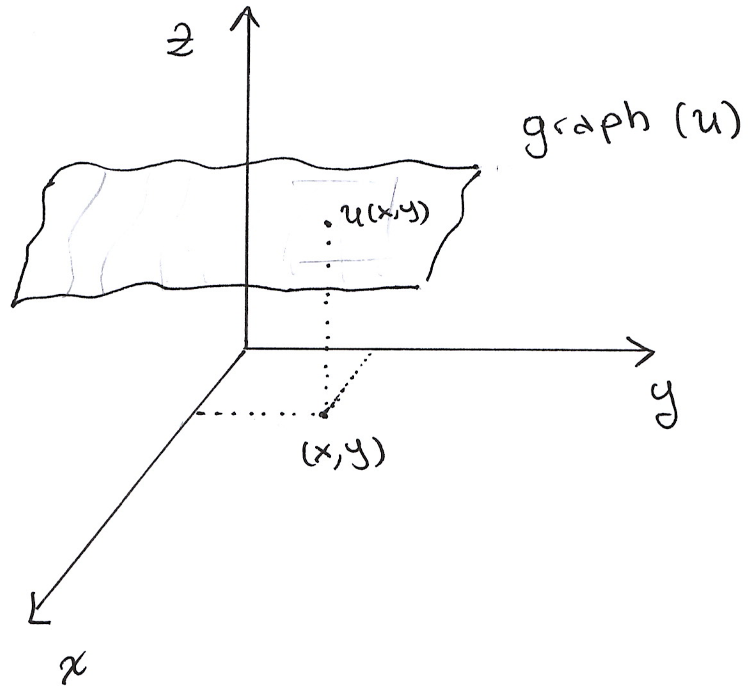


Fig 1: graph of the solution surface

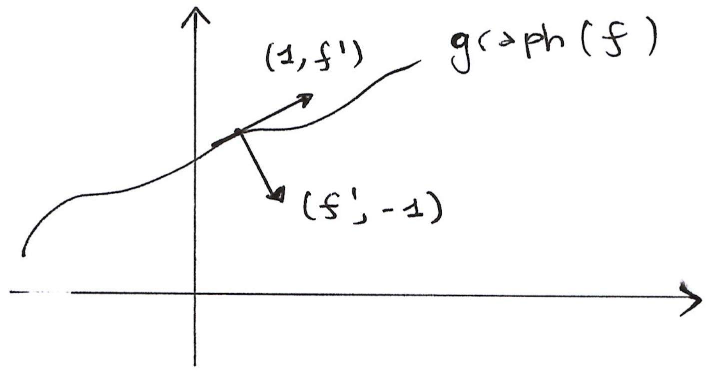


Fig 2: The tangent to the graph of a function $f : \mathbb{R} \to \mathbb{R}$ is given by the vector $(1, f')$

Therefore at each point the tangent plane to the surface graph(u) is the plane spanned by $(1, 0, u_x)$ and $(0, 1, u_y)$. Equivalently the vector $(u_x, u_y, -1)$ is orthogonal to this surface. Thus (2) can be thought as a pointwise relation between u and the tangent plane to the graph of u at the point $(x, y, u(x,y))$.

Quasilinear equations

(Section 2.2)

Quasilinear equations are nonlinear PDEs where the nonlinearity is confined to the unknown function u while the derivatives of u appear linearly.

The general form of a first order quasilinear PDE (in two variables) is the following:

$$(3) \qquad a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$$

a, b, c functions.

Linear equations are a particular case of quasilinear equations:

$$(4) \qquad a(x,y)u_x + b(x,y)u_y = c_0(x,y)u + c_1(x,y)$$

a, b, c_0, c_1 functions.

Example (2.1): Consider the first order PDE

$$u_x(x,y) = c_0 u(x,y) + c_1(x,y) \quad , \quad c_0 \text{ constant}.$$

No u_y appearing! $a = 1$, $b = 0$, $c_0 \in \mathbb{R}$

For each $y \in \mathbb{R}$ fixed this is a first order ODE, that can be rewritten as

$$\left[u_x(x,y) - c_0 u(x,y)\right] \underbrace{e^{-c_0 x}}_{\text{integrating factor}} = c_1(x,y)\, e^{-c_0 x}$$

or equivalently:

$$\frac{\partial}{\partial x}\left(u(x,y)\, e^{-c_0 x}\right) = c_1(x,y)\, e^{-c_0 x}$$

Integrating both sides over an interval of the form $[x_0(y), x]$ we have:

$$u(x,y)\, e^{-c_0 x} - u(x_0(y),y)\, e^{-c_0 x_0(y)} = \int_{x_0(y)}^{x} e^{-c_0 \xi} c_1(\xi, y)\, d\xi$$

therefore

$$u(x,y) = e^{c_0 x} \left[u(x_0(y),y)\, e^{-c_0 x_0(y)} + \int_{x_0(y)}^{x} e^{-c_0 \xi} c_1(\xi,y)\, d\xi \right]$$

Given the value on one point we get the value on a line

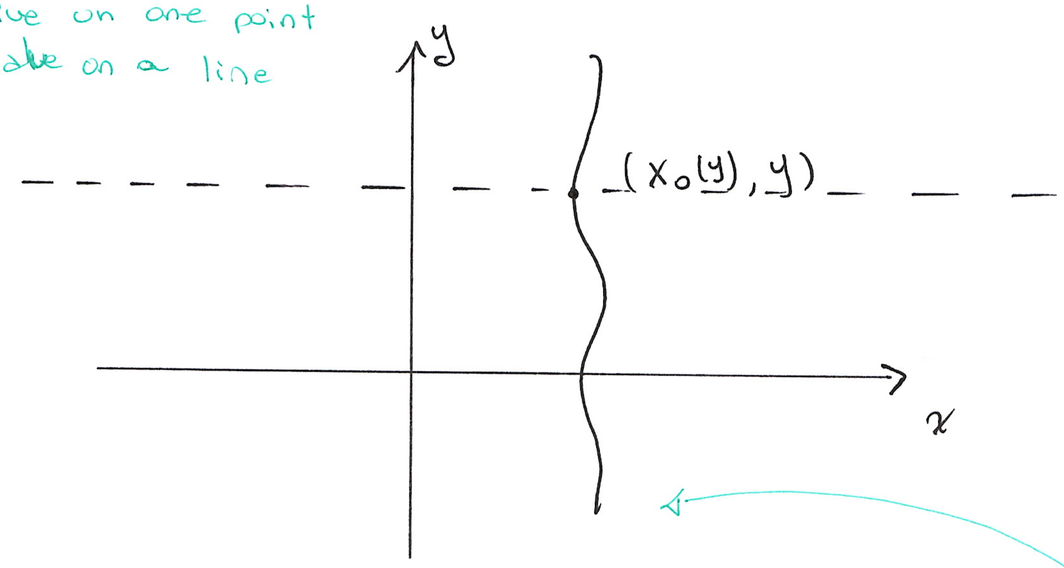


Fig 3: once we prescribe the value of u on this curve (for example at the intersection with the dotted curve) we can reconstruct the value of u everywhere on the dotted line.

Depending by the initial condition we may have one solution, no solutions or infinitely many solutions.
Let us now consider some possible initial conditions:

- $u(0,y) = y$ for all $y \in \mathbb{R}$. Then in this case we set $x_0(y) = 0$ and $u(x_0(y), y) = y$
  Then the solution of the equation is given by

$$u(x,y) = e^{c_0 x} \left[\int_0^x e^{-c_0 \xi} c_1(\xi, y)\, d\xi + y \right].$$

∃! SOLUTION (two conditions out of three for well-posedness are verified for sure!)

- Assume that $c_1 \equiv 0$ so that the general solution is given by

$$(5) \quad u(x,y) = e^{c_0 x}\Big[\underbrace{u(x_0(y),y)\, e^{-c_0 x_0(y)}}_{T(y)}\Big].$$

$T(y)$ determined by the initial conditions

$$u(x,y) = e^{c_0 x}\, T(y)$$

If we now prescribe as initial condition $u(x,0) = a x$, $a \in \mathbb{R}$, $a \neq 0$, then $T(y)$ should satisfy $T(0) = u(x,0)\, e^{-c_0 x} = a x\, e^{-c_0 x}$

which is impossible. ⇝ NO SOLUTION

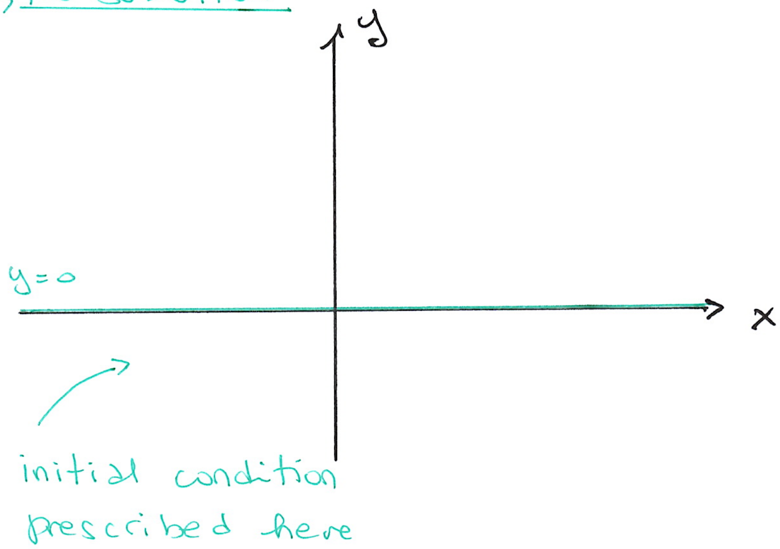
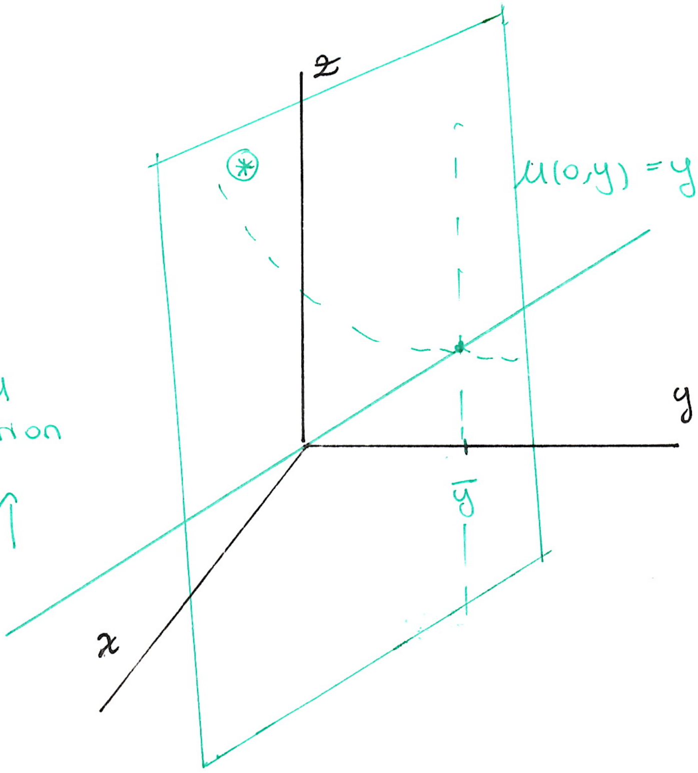


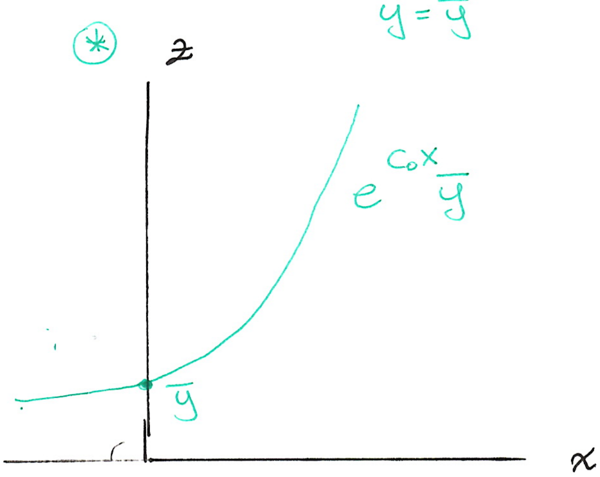
Fig 4: Once we prescribe the value of u at one point on the curve $y=0$ then the formula (5) gives the value of u at all other points on the curve $y=0$. If the value prescribed on $y=0$ is compatible with (5) then we have infinitely many solutions, otherwise no solutions.

Figures illustrating the case of 1 solution in 3D and of no solutions



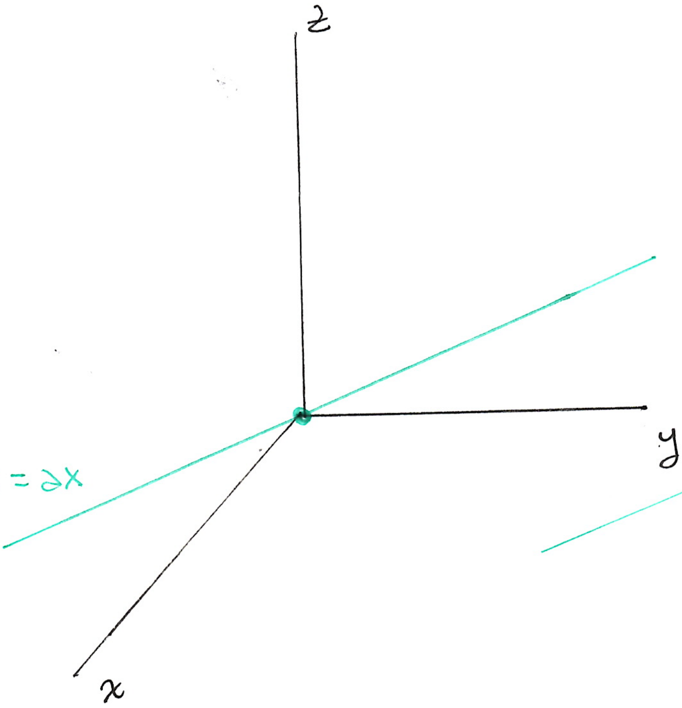
Initial condition
↑

this corresponds to the projection on the section $y = \bar{y}$

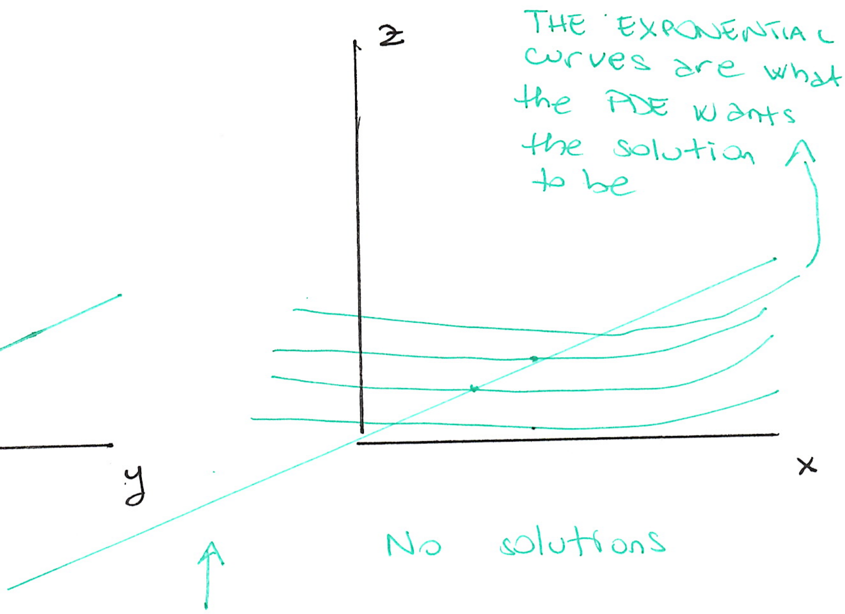


Here the initial datum is only 1 point in the plane $y = \bar{y} \Rightarrow \exists!$ solution

Section $y=0$



$u(x,0) = ax$



No solutions

↑
INITIAL DATUM $\neq$ what the PDE gives

- We have seen a case in which the eq. has a unique solution and a case in which there was no solution at all.

Now let's see a case when the equation may have infinitely many solutions.

$$\begin{cases} u_x = c_0 u & \text{as before } (c_1 \equiv 0) \\ u(x,0) = e^{c_0 x} & \forall x \in \mathbb{R} \end{cases}$$

General formula for the solution: as before in (5).

$$u(x,y) = e^{c_0 x}\left[u(x_0(y),y)\cdot e^{-c_0 x_0}\right]$$

Therfore, if we take $y=0$ we have

if it is 1 then we are done!

$$e^{c_0 x} \overset{\text{I. cond.}}{=} u(x,0) \overset{\text{General formula}}{=} e^{c_0 x}\, \underbrace{u(x_0(0),0)\, e^{-c_0 x_0(0)}}\,,$$

thus in order to ensure $u(x,0) = e^{c_0 x}$ we need to impose

$$u(x_0(0),0)\, e^{-c_0 x_0(0)} = 1.$$

This is an undetermined problem that has infinitely many solutions. Indeed we can take any curve $y \mapsto (x_0(y), y)$ and choose any value we want for $u(x_0(y),y)$ provided that $u(x_0(0),0) = e^{x_0(0)}$.

To give some example, take as curve $y \mapsto (0,y)$ $(x_0(0)=0)$ and set $u(0,y) = 1 + Ay^2$. Then

$$u(x,y) = e^{c_0 x}(1 + Ay^2) \quad \text{is a solution for}$$

every $A \in \mathbb{R}$.

Conclusion of example 2.1 :

If we consider the PDE

$$u_x(x,y) = c_0\, u(x,y) \qquad (c_1 \equiv 0)$$

than the general formula for a solution is:

$$u(x,y) = e^{c_0 x}\, u(x_0(y), y)\, e^{-c_0 x_0(y)}$$

so for all y we take $x_0(y) \in \mathbb{R}$ and define a curve $(x_0(y), y)$. Then the value of u at each point $(x_0(y), y)$ is propagated along orizontal lines (Fig 3).

- In the first case the initial condition was $y = u(0,y)$ and we have one solution (unique).

- In the second case $u(x,0) = ax$, $a \neq 0$ is not compatible with the PDE and we have no solutions.

- In the third case the initial condition $u(x,0) = e^{c_0 x}$ is compatible with the PDE but this condition leaves us "too much choice" because apart from $(0,0)$ where $u(0,0) = 1$, we have complete freedom and we have infinitely many possible solutions for the PDE.

The moral of the story: Boundary conditions and initial conditions are very important. We need to be careful to impose appropriate conditions in order to obtain a well posed PDE.

The Method of Characteristics

1. First order linear PDEs (Section 2.3)

Consider the general first order linear equation in two independent variables

(4) $a(x,y)u_x(x,y) + b(x,y)u_y(x,y) = c_0(x,y)u(x,y) + c_1(x,y)$

We want to assign the value of the solution u along a parametric curve and then "propagate" this value along the characteristic curves in order to "knit" the solution surface $u(x,y)$ using a one-parameter family of curves that intersect the initial curve.

So, given a curve $s \mapsto (x_0(s), y_0(s))$ we prescribe the value of u along such curve:

$$\tilde{u}_0(s) = u(x_0(s), y_0(s)) \quad , s \in \mathbb{R}.$$

Since the initial condition gives us some information about the solution, it can be represented as a curve on the solution surface. More precisely we consider the parametrized curve in $\mathbb{R}^3$

$$\Gamma : \Gamma(s) = (x_0(s), y_0(s), \tilde{u}_0(s)).$$

This is called initial curve.

Therefore equation $(\star)$ can be rewritten as

$$\underbrace{(a, b, c_0 u + c_1)}_{\mathbf{V}} \cdot (u_x, u_y, -1) = 0$$

scalar product

ortogonality condition

Since the vector $(u_x, u_y, -1)$ is orthogonal to the surface $(x, y, u(x,y)) = \mathrm{graph}(u)$, then the vector $\mathbf{V} := (a, b, c_0 u + c_1)$ must lie in the tangent plane to $\mathrm{graph}(u)$. Hence if we integrate this vector field (namely we consider the ODE $\dot{\mathbf{Z}} = \mathbf{V}(\mathbf{Z})$), then the curve $\underset{(x,y,z)}{\overset{\shortparallel}{\mathbf{Z}}}$ will be contained inside the surface $\mathrm{graph}(u)$.

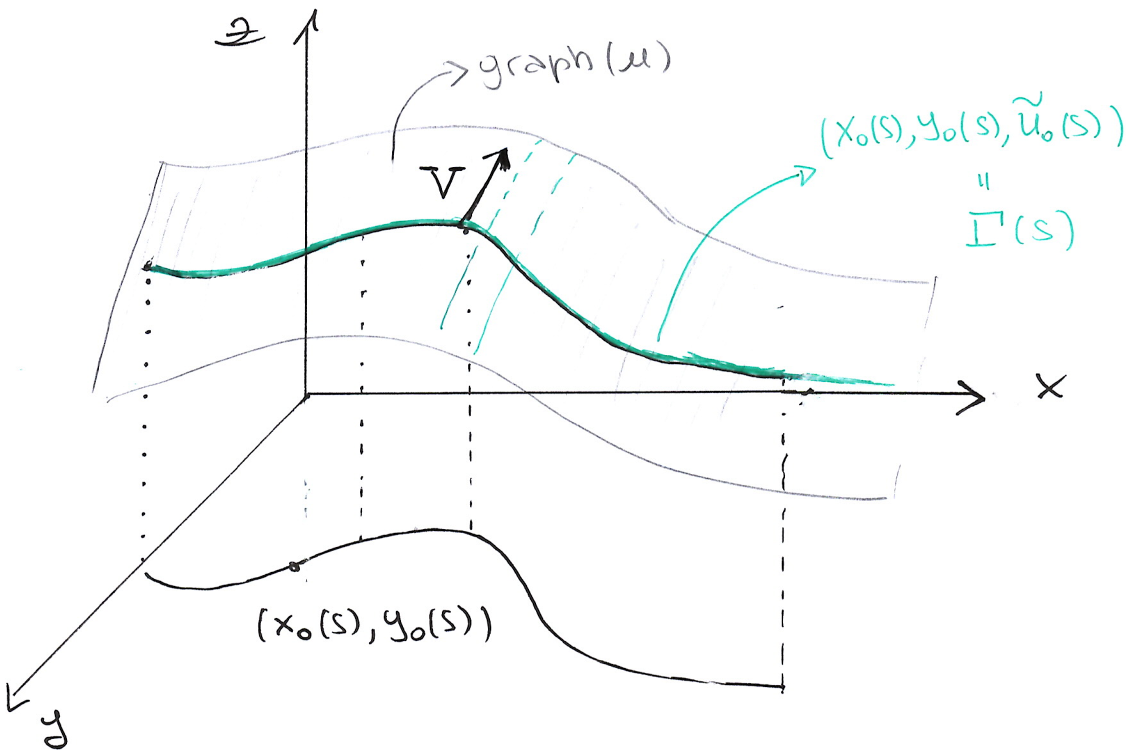


Fig 5: The initial curve $\Gamma(s)$ and the construction of the solution surface

GOAL: Find u such that $\Gamma \subseteq \mathrm{graph}(u)$ and u solves the PDE.

With this geometric consideration in mind we rewrite (*) as

$$(a, b, c_0 u + c_1)\cdot(u_x, u_y, -1) = 0$$

i.e. $\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{\tilde{u}}\end{pmatrix} = V\begin{pmatrix}x\\ y\\ \tilde{u}\end{pmatrix}$

and consider the ODE system:

ODE system

$$\begin{cases} \dfrac{dx(t,s)}{dt} = a(x(t,s), y(t,s)) \\[2mm] \dfrac{dy(t,s)}{dt} = b(x(t,s), y(t,s)) \\[2mm] \dfrac{d\tilde{u}(t,s)}{dt} = c_0(x(t,s), y(t,s)).\tilde{u}(t,s) + c_1(x(t,s), y(t,s)). \end{cases}$$

these ODEs are autonomous = no explicit dependence upon the parameter t

The solutions of this system of ODEs are called characteristic equations.

To find the characteristics $(x(t,s), y(t,s), \tilde{u}(t,s))$ we need to couple our system with suitable initial conditions. Recalling that the initial curve $\Gamma(s)$ must be contained in graph(u) we have

Initial conditions

$$\begin{cases} x(0,s) = x_0(s) \\ y(0,s) = y_0(s) \\ \tilde{u}(0,s) = \tilde{u}_0(s) \end{cases}$$

we require the initial point to lie on the initial curve Γ: each curve $(x(t,s), y(t,s), \tilde{u}(t,s))$ emanates from a different point $\Gamma(s)$.

Solving the above system of ODEs together with the initial conditions, we obtain a parametrized representation of the solution surface graph(u) in the variables (t,s). Then we shall re-express (whenever possible) the surface in terms of (x,y):

$$\boxed{\tilde{u}(t,s) = u(x(t,s), y(t,s))}$$

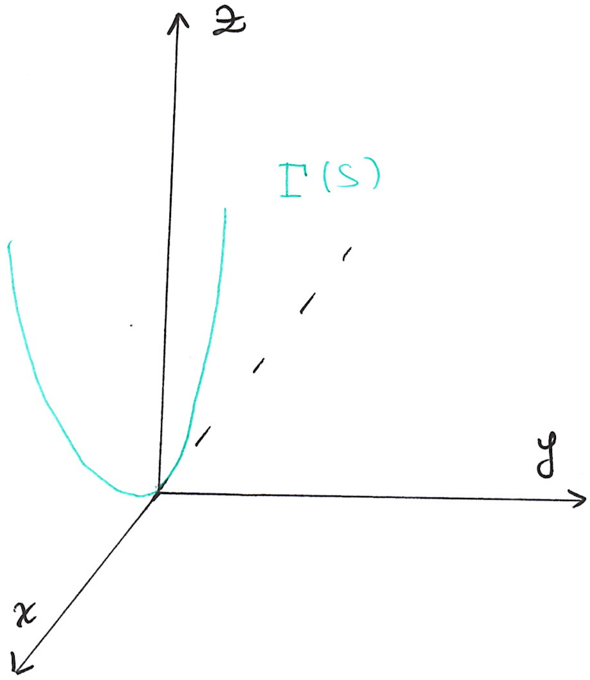
← very important

Example 2.2

Consider the Cauchy problem (i.e. PDE + I.C.)

$$(CP)\begin{cases} u_x + u_y = 2 & (PDE) \\ u(x,0) = x^2 & (IC) \end{cases}$$

Fig 6: $\Gamma(s) = (s, 0, s^2) \leftarrow$ Initial curve



We parametrise the initial condition as

$$(*)\begin{cases} x_0(s) = s \\ y_0(s) = 0 \\ \tilde{u}_0(s) = s^2 \end{cases}$$

and we obtain the following system:

$$\begin{cases} \frac{d}{dt} x(t,s) = a(x(t,s), y(t,s)) = 1 \\ \frac{d}{dt} y(t,s) = b(x(t,s), y(t,s)) = 1 \\ \frac{d}{dt} \tilde{u}(t,s) = c_0(x(t,s), y(t,s)).\tilde{u}(t,s) + c_1(x(t,s), y(t,s)) = 2 \end{cases}$$

together with the initial conditions $(*)$.

Therefore the characteristic curves are given by:

$$\begin{cases} x(t,s) = s + t \\ y(t,s) = t \\ \tilde{u}(t,s) = s^2 + 2t \end{cases}$$

Remark: There is no unique way to parametrise the initial condition. We could have defined $\Gamma(s) = (s^3, 0, s^6)$ and this gives the same initial conditions.

The parametrized solution surface is then given by the relation

$$u(x(t,s), y(t,s)) = \tilde{u}(t,s) = s^2 + 2t.$$

Since we are looking for a solution in (x,y) coordinates, we have to find the inverse map $(t,s) \mapsto (x,y)$ to find $u(x,y)$.

In this case it is very easy:

$$\begin{cases} x(t,s) = s + t \\ y(t,s) = t \end{cases} \rightsquigarrow \begin{cases} y = t \\ s = x - t = x - y \end{cases}$$

Hence the solution to the PDE is given by:

$$u(x,y) = s^2 + 2t = (x-y)^2 + 2y$$

Summary:

- First-order PDEs relate the solution surface to its tangent plane
- They can be solved using the method of characteristics
- The parametrized initial condition is called initial curve, which is used to solve the characteristic equations
- The obtained characteristic curves form the solution surface