Lecture 3 7/10/2019 3.0

Quick "How to":

The method of characteristics for quasilinear PDEs.

Consider the Cauchy problem:

$$\begin{cases} a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u), & (x,y)\in\Omega\subseteq\mathbb{R}^2 \quad \text{PDE} \\ u(x,y) = u_0(x,y) & (x,y)\in\gamma \quad \text{I.C.} \end{cases}$$

- Choose a parametrisation of initial curve:

$\Gamma(s) = (x_0(s), y_0(s), \tilde{u}_0(x_0(s), y_0(s)))$, $(x_0(s), y_0(s))\in\gamma$.

STEP 1: Solve the following 1-parameter family of ODEs to find $x(t,s)$, $y(t,s)$, $\tilde{u}(t,s)$:

$$\frac{d}{dt}x(t,s) = a(x(t,s), y(t,s), \tilde{u}(t,s))$$

$$\frac{d}{dt}y(t,s) = b(x(t,s), y(t,s), \tilde{u}(t,s))$$

$$\frac{d}{dt}\tilde{u}(t,s) = c(x(t,s), y(t,s), \tilde{u}(t,s))$$

With Initial Conditions $x(0,s) = x_0(s)$, $y(0,s) = y_0(s)$, $\tilde{u}(0,s) = \tilde{u}_0(s)$. In this way $\underbrace{\tilde{u}(t,s) = u(x(t,s), y(t,s))}_{\text{DESIRED SOLUTION}}$.

The curves $t\mapsto (x(t,s), y(t,s))$ are called characteristics (or projected characteristics).

STEP 2: Invert the function $(t,s)\mapsto(x(t,s), y(t,s))$ to write t and s in terms of x and y.

(whenever possible)

STEP 3: Define $u(x,y) = \tilde{u}(t(x,y), s(x,y))$.

- In this lecture we continue our discussion about the method of characteristics for first-order quasilinear PDEs.

- We discuss some conditions that guarantee local existence and uniqueness

- The question is: under which conditions there exists a unique integral surface for

$$(1) \quad a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$$

that contains the initial curve Γ?

Recall about the MoC (Sec. 2.3 after example 2.2)

General form of a Cauchy problem where the PDE is quasi-linear, first-order, in two real variables, and the initial condition is represented by a curve $\Gamma \subseteq \mathbb{R}^3$ (initial curve):

Cauchy Problem
$$\begin{cases} a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \longrightarrow \text{PDE} \\ \Gamma(s) = (x_0(s), y_0(s), u_0(s)) \longrightarrow \text{Initial curve} \end{cases}$$

Note that the curve Γ is often implicitly given by the "boundary condition" for u, for instance

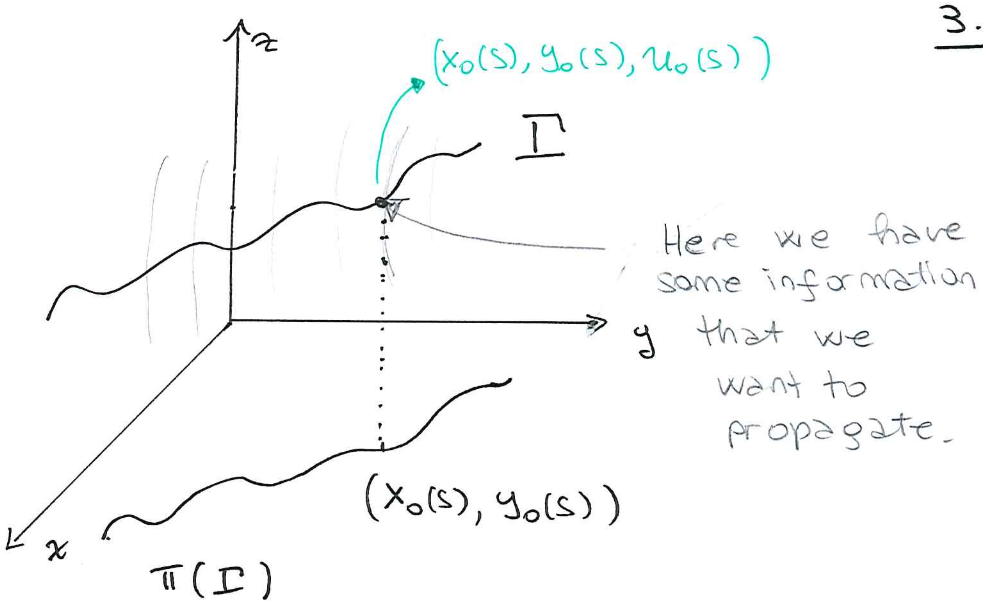
$$u(x,0) = x^2.$$

In order to apply the M.o.C. the curve Γ needs to be parametrized: in Example 2.2 it was

$$\Gamma(s) = (s, 0, s^2), \quad s \in \mathbb{R}.$$

Fig 7: The initial curve Γ

Note, $\pi(\Gamma)$ coincides with γ at page 3.0



To solve our Cauchy problem we need to solve the characteristic equations using the point we selected on Γ as an initial condition for the ODEs.

$$\underbrace{\begin{cases} \frac{d}{dt} x(t,s) = a(x(t,s), y(t,s), \tilde{u}(t,s)), \\ \frac{d}{dt} y(t,s) = b(x(t,s), y(t,s), \tilde{u}(t,s)), \\ \frac{d}{dt} \tilde{u}(t,s) = c(x(t,s), y(t,s), \tilde{u}(t,s)), \end{cases}}_{\text{ODEs}} \quad \underbrace{\begin{array}{l} x(0,s) = x_0(s) \\ y(0,s) = y_0(s) \\ \tilde{u}(0,s) = u_0(s) \end{array}}_{\text{Initial conditions}}$$

Assuming that the coefficients of the ODEs are smooth (a and b have to be C^1) we can apply the Cauchy-Lipschitz theorem for ODEs that guarantees local in time existence and uniqueness of the solution.

So, for each s there exists some time interval $I_s \subseteq \mathbb{R}$ such that the solution $t \mapsto (x(t,s), y(t,s), \tilde{u})$ exists uniquely for all $t \in I_s$.

Once solved the ODE system, we have an expression for $\tilde{u}$ in the variables (t,s).
The fundamental relation between $\tilde{u}(t,s)$ and $u(x,y)$ (the desired solution) is given by:

$$\tilde{u}(t,s) = u(x(t,s), y(t,s)).$$

Now some difficulties may arise in the inversion of the transformation from (t,s) to (x,y) because the mapping

$$\begin{cases} x = x(t,s) \\ y = y(t,s) \end{cases}$$

may not be invertible. Thanks to the implicit function theorem we know tha this map is locally invertible if

$$(*) \qquad \boxed{\det \begin{pmatrix} \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{pmatrix} \neq 0.} \longrightarrow$$

TRANSVERSALITY CONDITION
(Def 2.9 in PR)

Note that, if we consider the above expression at $(0,s)$ the condition $(*)$ becomes:

$$\boxed{\det \begin{pmatrix} \overset{\frac{d}{dt}x}{a(x_0(s), y_0(s), u_0(s))} & \overset{\frac{d}{dt}y}{b(x_0(s), y_0(s), u_0(s))} \\ \frac{d}{ds} x_0(s) & \frac{d}{ds} y_0(s) \end{pmatrix} \neq 0}$$

because $x(0,s) = x_0(s)$, $y(0,s) = y_0(s)$, $\frac{d}{dt}x = a$, $\frac{d}{dt}y = b$.

Asking that the transversality condition is verified means that the vectors (a, b) and $\left(\frac{d}{ds} x_0(s), \frac{d}{ds} y_0(s)\right)$ are transverse. (Fig 8)

(Transversality can be seen as the "opposite" of tangency)

Since $(a(x_0(s), y_0(s), u_0(s)), b(x_0(s), y_0(s), u_0(s)))$ is the tangent vector to the characteristic $t \mapsto (x(t,s), y(t,s))$ at $t=0$, while $\left(\frac{d}{ds} x_0(s), \frac{d}{ds} y_0(s)\right)$ is the tangent vector to $\pi(\Gamma)$ at s, the transversality condition means that $\pi(\Gamma)$ and $t \longmapsto (x(t,s), y(t,s))$ are transverse at $t=0$ (to be intended as non tangen=tial).

Another way to say this is that the Jacobian (see transversality condition) vanishes at some point if and only if the vectors (a, b) and $\left(\frac{d}{ds} x_0(s), \frac{d}{ds} y_0(s)\right)$ are linearly dependent.

So far we discussed local problems (i.e. problems that forbid local existence of a solution).

We can also encounter obstacles to global existence.

Global existence (existence of a solution in all of the domain of interest) can fail for several reasons:

(i) In general ODEs only have local solutions, and solutions can blow up in finite time. Similarly solutions of Cauchy problem can blow up if you move far enough away from Γ

ii) If the characteristics $t \mapsto (x(t,s), y(t,s))$ intersect the Cauchy (a.k.a. initial) curve Γ more than once, then global existence may fail. This is because the characteristic eq. is well-posed for a single initial condition. Think about the fact that a characteristic curve "carries" with it a charge of information from its intersection point with Γ. If a characteristic curve intersects Γ more than once these two "information charges" might be in conflict. (Fig 9)

iii) If the vector field (a, b) vanishes at some point, then the corresponding PDE may only have a solution outside of a neighbourhood of this point

iv) If the characteristics intersect within the domain of interest, then existence can break down at the intersection points.

Alternatevely it may happen that the characteristic curves (as curves in $\mathbb{R}^3$) project all onto the same curve in the (x,y)-plane.

In this case

(A) either the characteristic curves coincide ($\forall$ s, s', γ_s and $\gamma_{s'}$ parametrise the same curve) in which case there are infinitely many solutions

(B) these curves do not coincide, which means that graph (u) should take different values above $\pi(\Gamma)$ which is impossible.

Later we will see the exact statement of the existence theorem. Before that, let's see some other example.

Example 2.3 (Section 2.4)

Consider the following Cauchy Problem

$$\begin{cases} u_x = 1 \\ u(0,y) = g(y) \end{cases}, \quad g \text{ generic function}$$

We parametrize the curve by choosing

$\Gamma(s) = (0, s, g(s))$.

The characteristic equations are then given by:

$$\begin{cases} \frac{d}{dt} x(t,s) = 1, \quad x(0,s) = 0 \quad \Rightarrow \quad x(t,s) = t \\ \frac{d}{dt} y(t,s) = 0, \quad y(0,s) = s \quad \Rightarrow \quad y(t,s) = s \\ \frac{d}{dt} \tilde{u}(t,s) = 1, \quad \tilde{u}(0,s) = g(s) \Rightarrow \tilde{u}(t,s) = g(s) + t \end{cases}$$

The relation between the variables x and y and the variables t and s is easy:

$$t = x$$
$$s = y$$

Therefore $\tilde{u}(t,s) = u(x(t,s), y(t,s))$ give us the solution $u(x,y) = g(y) + x$.

(Fig 10)

Fig 8 :

the projection of a characteristic curve crossing $\pi(\Gamma)$ transversally

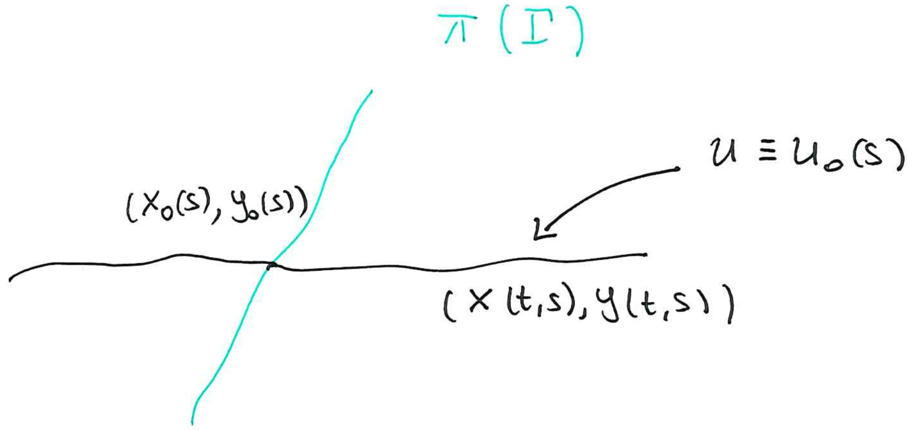


Fig 9 : projection of a characteristic curve crossing $\pi(\Gamma)$ twice. The value of u at $(x_0(s'), y_0(s'))$ may not be uniquely defined since it should both be equal to $u_0(s')$ and to $u(x(t',s), y(t',s))$.

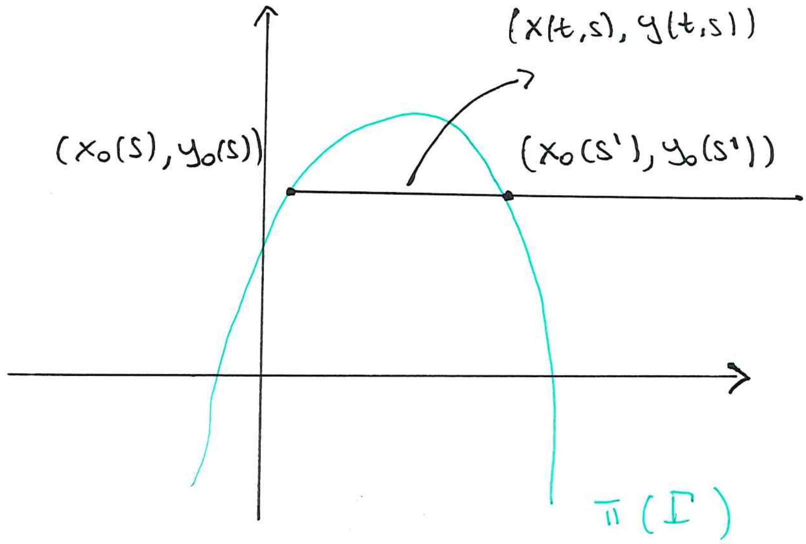
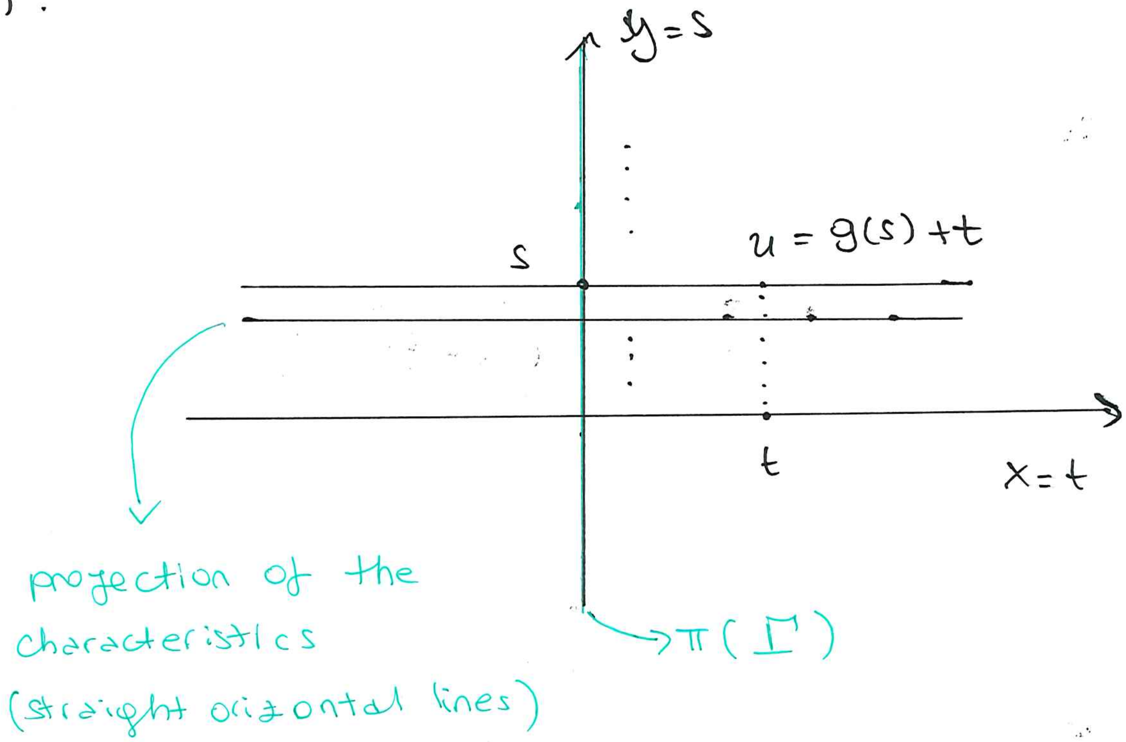


Fig 10 :
Example 2.3



Example 2.3 continuation: let us now consider the same PDE with a different initial condition:

$$\begin{cases} u_x = 1 \\ u(x,0) = h(x) \end{cases}$$

Parametrization of the initial curve:
$\Gamma(s) = (s, 0, g(s))$

We have the following characteristic equations:

$$\begin{cases} \frac{d}{dt} x(t,s) = 1, \quad x(0,s) = s \implies x(t,s) = t + s \\ \frac{d}{dt} y(t,s) = 0, \quad y(0,s) = 0 \implies y(t,s) = 0 \\ \frac{d}{dt} \tilde{u}(t,s) = 1, \quad \tilde{u}(0,s) = h(s) \implies \tilde{u}(t,s) = h(s) + t \end{cases}$$

Now, however, the relation $(x(t,s), y(t,s))$ cannot be inverted

This can be seen evaluating the determinant:

$$\det \begin{pmatrix} \overset{\to a}{\frac{\partial x}{\partial t}} & \overset{\to b}{\frac{\partial y}{\partial t}} \\ \underset{\to (x_0)_s}{\frac{\partial x}{\partial s}} & \underset{\to (y_0)_s}{\frac{\partial y}{\partial s}} \end{pmatrix} = \det \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = 0$$

Note that the projection of the initial curve is the x axis, but this is also the projection of a characteristic curve. In the case where $h(x) = x + c$, $c \in \mathbb{R}$ we obtain $\tilde{u}(t,s) = s + t + c$. Then it is not necessary to invert the mapping $(x(t,s), y(t,s))$ because $u = x + c + f(y)$ is a solution for every differentiable

$f(y)$ that vanishes at the origin. But for any other choice of h the problem has no solutions.

Example 2.5 : Consider the following Cauchy problem:

$$\begin{cases} u_x + u_y + u = 1 \\ u = \sin x \quad \text{on } y = x + x^2, \; x > 0 \end{cases}$$

Initial curve

$$\Gamma(s) = (x_0(s), y_0(s), \tilde{u}_0(s)) = (s, s+s^2, \sin(s))$$

The characteristic equations:

$$\begin{cases} \frac{d}{dt} x(t,s) = 1, \quad x(0,s) = s \quad \Rightarrow x(t,s) = s+t \\ \frac{d}{dt} y(t,s) = 1, \quad y(0,s) = s+s^2 \Rightarrow y(t,s) = s+s^2+t \\ \frac{d}{dt} \tilde{u}(t,s) = 1-\tilde{u}, \quad \tilde{u}(0,s) = \sin(s) \end{cases}$$

Solving the equation for $\tilde{u}$

integrating factor

$$\frac{d}{dt}\tilde{u} + \tilde{u} = 1 \Rightarrow \left(\frac{d}{dt}\tilde{u} + \tilde{u}\right)e^t = e^t$$

$$\Rightarrow \frac{d}{dt}\left(\tilde{u}\, e^t\right) = e^t$$

Therefore we have:

$$\tilde{u}(t,s)e^t - \tilde{u}(0,s) = \int_0^t e^{\tau}\, d\tau = e^t - 1$$

$$\tilde{u}(t,s) = e^{-t}\left[\tilde{u}(0,s) + e^t - 1\right] = e^{-t}\sin(s) + 1 - e^{-t}.$$

From the relations $x = s+t$ and $y = s+s^2+t$ we get

$y = s^2 + x$. Then, since $s^2 > 0$ it must hold $y > x$.

So $y = s^2 + x \Rightarrow \sqrt{y-x} = s$.

Going back to the relation

$x = s + t$ we get

$$t = x - s = x - \sqrt{y-x} \, .$$

Plugging this into the equation for u we obtain:

$$u(x,y) = e^{-x+\sqrt{y-x}} \left[\sin(\sqrt{y-x}) - 1 \right] + 1$$

Remark: compute the Jacobian for the initial curve:

$$\det \begin{pmatrix} 1 & 1 \\ 1 & 1+2s \end{pmatrix} = 2s \neq 0 \quad \forall s \neq 0$$

therefore we could expect the existence of a unique solution at each point where $s \neq 0$.

Example 2.6: Consider the PDE:

$$\begin{cases} -y u_x + x u_y = u \\ u(x,0) = \varphi(x) \end{cases}$$

$$\Gamma(s) = (x_0(s), y_0(s), \tilde{u}_0(s)) = (s, 0, \varphi(s))$$

$$\begin{cases} \frac{d}{dt} x(t,s) = -y \, , & x_0(s) = s \\ \frac{d}{dt} y(t,s) = +x \, , & y_0(s) = 0 \\ \frac{d}{dt} \tilde{u}(t,s) = \tilde{u}(t,s) \, , & \tilde{u}_0(s) = \varphi(s) \end{cases}$$

Let us examine the transversality condition:

$$\det \begin{pmatrix} 0 & s \\ 1 & 0 \end{pmatrix} = -s \quad \checkmark$$

ok $\forall s \neq 0$

We have to solve:

$$\begin{cases} \frac{d}{dt} x = -y \\ \frac{d}{dt} y = x \end{cases}$$

Note that $\frac{d^2}{d^2t} x = -\frac{d}{dt} y = -x$ and analogously

$$\frac{d^2}{dt^2} y = \frac{d}{dt} x = -y.$$

Hence:

$$\begin{cases} \frac{d^2}{d^2t} x = -x \Rightarrow x(t,s) = f_1(s)\cos(t) + f_2(s)\sin(t) \\ \frac{d^2}{d^2t} y = -y \Rightarrow y(t,s) = g_1(s)\cos(t) + g_2(s)\sin(t) \\ \frac{d}{dt}\tilde{u} = \tilde{u} \Rightarrow \tilde{u}(t,s) = \tilde{u}(0,s)e^t \Rightarrow \tilde{u}(t,s) = \varphi(s)e^t. \end{cases}$$

Using that $x(0,s) = s$, $y(0,s) = 0$ and $\frac{d}{dt} x(0,s) = -y(0,s) = 0$, $\frac{d}{dt} y(0,s) = x(0,s) = s$ we obtain

$x(t,s) = s\cos(t)$, $y(t,s) = s\sin(t)$

$x(0,s) = f_1(s) = s$, $y(0,s) = g_1(s) = 0$

$\frac{d}{dt} x(0,s) = -f_1(s)\underbrace{\sin(0)}_{0} + f_2(s)\underbrace{\cos(0)}_{1} = f_2(s) = 0$

$\frac{d}{dt} y(0,s) = -g_1(s)\sin(0) + g_2(s)\cos(0) = g_2(s) = s$

$\Rightarrow x(t,s) = f_1(s)\cos(t) = s\cos(t)$

$y(t,s) = g_2(s)\sin(t) = s\sin(t)$ ✓

If we assume $s>0$ we note that s and t act like polar coordinates r and φ, so we can invert the relations

$$\begin{cases} x(t,s) = s\cos(t) \\ y(t,s) = s\sin(t) \end{cases}$$

as :

$$\begin{cases} s = \sqrt{x^2+y^2} \\ t = \arctan\left(\frac{y}{x}\right) \end{cases}$$

which gives us

$$\underline{u(x,y) = \varphi\left(\sqrt{x^2+y^2}\right) e^{\arctan\left(\frac{y}{x}\right)}}$$

Fig 11:

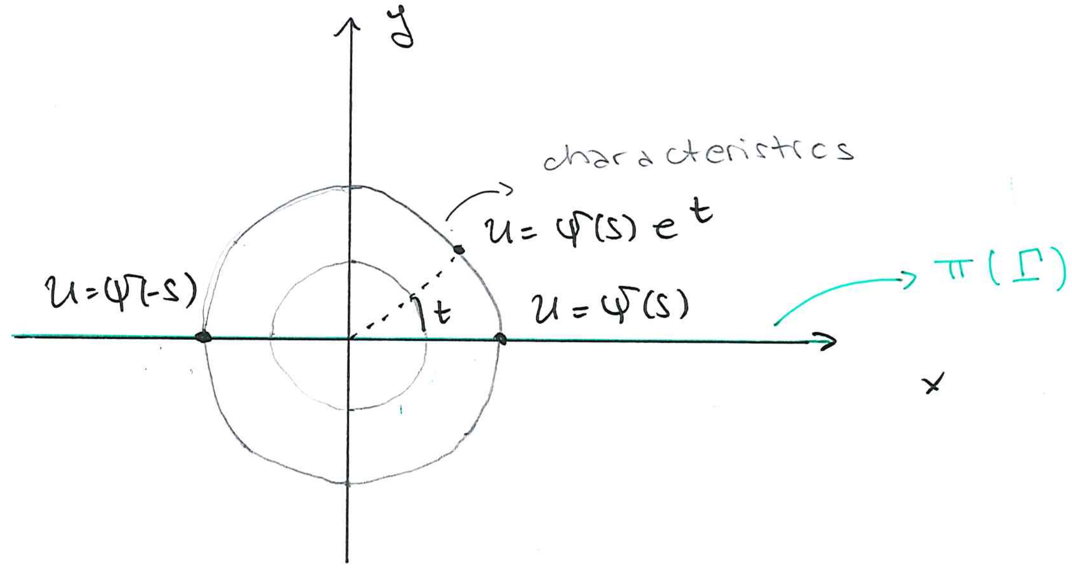


Fig The characteristics form a one-parameter family of circles around the origin. Therefore each of them intersect the projection of the initial curve $\pi(\Gamma)$ twice. Moreover the Jacobian vanishes at the origin. We have a unique solution because, choosing the sign $+$ for $\sqrt{x^2+y^2}$ we reduced ourselves at the case $s>0$, where we have only one intersection.

The existence and uniqueness theorem for the M.o.C. (Section 2.5)

As we have seen in the previous sections the existence and uniqueness of solutions is a delicate issue. In fact:

(1) the projection of the characteristics may not be transversal to the initial curve, in which case we will not be able to express t and s in terms of x and y.

(2) the projection of a characteristic may intersect $\Pi(\Gamma)$ more than once, in which case the value of u may not be uniquely defined.

We have the following local existence and uniqueness result:

Theorem 2.10: Consider the Cauchy problem

$$\begin{cases} a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \\ \Gamma(s) = (x_0(s), y_0(s), u_0(s)) \end{cases}$$

Assume that there exists $s_0 \in \mathbb{R}$ for which the transversality condition holds at $(0, s_0)$, i.e.:

$$(*) \quad \det \begin{pmatrix} \frac{\partial x}{\partial t}(0,s_0) & \frac{\partial y}{\partial t}(0,s_0) \\ \frac{\partial x}{\partial s}(0,s_0) & \frac{\partial y}{\partial s}(0,s_0) \end{pmatrix} \neq 0 .$$

then there exists a unique solution u of the Cauchy problem defined in a neighborhood of $(x_0(s_0), y_0(s_0))$.

Idea of the proof:

We first solve the characteristic equations for s close to s_0:

$$\begin{cases} \frac{d}{dt} x(t,s) = a(x(t,s), y(t,s), \tilde{u}(t,s)), & x(0,s) = x_0(s) \\ \frac{d}{dt} y(t,s) = b(x(t,s), y(t,s), \tilde{u}(t,s)), & y(0,s) = y_0(s) \\ \frac{d}{dt} \tilde{u}(t,s) = c(x(t,s), y(t,s), \tilde{u}(t,s)), & \tilde{u}(0,s) = u_0(s) \end{cases}$$

By existence and uniqueness for ODEs we know that there exists a unique solution $(x(t,s), y(t,s), \tilde{u}(t,s))$ defined for (t,s) close to $(0, s_0)$.

Thanks to the transversality condition $(*)$ we know that we can apply the implicit function theorem and the map
$$\begin{cases} x = x(t,s) \\ y = y(t,s) \end{cases}$$

is invertible close to $(0, s_0)$.

So this allow us to find a formula for u in a neighborhood of $(x(0,s_0), y(0,s_0)) = (x_0(s_0), y_0(s_0))$.

Summary:

- The M.o.C. can be used to solve 1st order quasilinear PDEs
- The initial condition is described by the initial curve
- The characteristics are expressed in terms of (t,s) instead of (x,y).
- We need to express the solution for u in terms of (x,y) but the map $(t,s) \mapsto (x,y)$ may not be invertible.