Lecture 4

Recall from lecture 3:

Last time we discussed some conditions that guarantee local existence and uniqueness for the following Cauchy problem:

$$\begin{cases} a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \\ \Gamma(s) = (x_0(s), y_0(s), u_0(s)) \end{cases}$$

M.o.C
$$\begin{cases} x_t(t,s) = a \\ y_t(t,s) = b \\ \tilde{u}_t(t,s) = c \end{cases} \quad + \quad \begin{matrix} x(0,s) = x_0(s) \\ y(0,s) = y_0(s) \\ \tilde{u}(0,s) = u_0(s) \end{matrix}$$

$$\tilde{u}(t,s) = u(x(t,s), y(t,s))$$

Now some issue may arise in the inversion of the transformation from (t,s) to (x,y) because the mapping

$$(*) \quad \begin{cases} x = x(t,s) \\ y = y(t,s) \end{cases}$$

may not be invertible.

We need to check the transversality condition at $(0,s)$:

$$\det \begin{pmatrix} a(x_0(s), y_0(s), u_0(s)) & b(x_0(s), y_0(s), u_0(s)) \\ \frac{d}{ds} x_0(s) & \frac{d}{ds} y_0(s) \end{pmatrix} \neq 0$$

If the transversality condition is verified we can invert $(*)$ in a neighborhood of $(0,s)$.

Conservation laws and Shock waves

(Section 2.7 PR)

- In this lecture we will study an important class of first order PDEs called $\underbrace{\text{conservation laws}}$.

  PDEs that prescribe conserved quantities such as: mass, electric charge, cars (traffic dynamics), people (crowd dynamics).

- We will see some examples (Burgers equation with various initial data)

- We can apply the M.o.C. to solve conservation laws but solutions of conservation laws may develop discontinuities even for smooth initial data

  $\Longrightarrow$ we need to introduce the notion of weak solution.

We recall that the transversality condition and the local existence theorem for first order quasilinear PDEs states that under suitable conditions, one can find local solutions to 1st order quasilinear PDEs using the M.o.C.

We will now see in some examples that, even if a classical solution ceases to exist, the phenomenon (say for example traffic flow) that we are modelling certainly does not.

Therefore we will broaden our definition of solution to allow us to make predictions about the phenomenon under study after the time when the classical solution ceases to exist.

- What are (scalar) conservation laws?

C.L. are PDEs describing the evolution of a conserved quantity. We will see some examples soon.
↳ for example: mass, energy, cars, people etc...

Definition: Scalar conservation laws for functions $u = u(x,y)$ of one spatial variable (x spatial variable, y time(*)) have the form:

$$(1) \quad \boxed{u_y + f(u)_x = 0}$$

where $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$, or equivalently

$$(2) \quad \boxed{u_y + c(u) u_x = 0}$$

where $c(u) = f'(u)$

We refer to f as the flux function.

- The easiest example of conservation law: the transport equation. (see equation 2.42 section 2.7 PR)

$$(3) \quad u_y + c\, u_x = 0$$

FIRST ORDER LINEAR PDE with $c \in \mathbb{R}$

In this case $c(u) = c \in \mathbb{R}$

(*) Why y for time? To be consistent with PR and not to have confusion with (t,s) in M.o.C.

Notice that $\underbrace{u_y + c u_x = 0}$ $\iff$ $\underbrace{u_y + (cu)_x = 0}$ therefore

structure as in (2) same structure as in (1)

$f(u) = cu$ flux, and $f'(u) = c(u) = c$.

If $u \geq 0$, u can represent the concentration of a pollutant in a river at time y and position x. The constant $c \in \mathbb{R}$ represents the velocity of the river: if $c > 0$ the flow is from left to right, if $c < 0$ the flow goes from right to left.

- The total amount of pollutant in $[a, b]$ is given by:

$$\int_a^b u(x,y)\,dx$$

Let us study the initial value problem (or Cauchy problem) for eq. (3):

velocity

$$\begin{cases} u_y + c u_x = 0 & (x,y) \in \mathbb{R} \times (0, +\infty) \quad \leftarrow \text{PDE} \\ u(x,0) = g(x) & x \in \mathbb{R} \quad \leftarrow \text{Initial condition} \end{cases}$$

, $g > 0$ concentration of the pollutant at time 0.

Let us now solve this problem using the M.o.C.

The equation $u_y + c u_x = 0$ has the form

$\tilde{a} u_x + \tilde{b} u_y = \tilde{c}$ with $\tilde{a} = c$, $\tilde{b} = 1$, $\tilde{c} = 0$.

The initial condition has the form

$$\Gamma(s) = (x_0(s), y_0(s), \tilde{u}_0(s)) = (s, 0, g(s))$$

Step 1 of the M.o.C :

The characteristic ODEs are :

$$\begin{cases} \dfrac{dx(t,s)}{dt} = c, & x_0(s) = s \Rightarrow \underline{x(t,s) = ct+s} \\ \dfrac{dy(t,s)}{dt} = 1, & y_0(s) = 0 \Rightarrow \underline{y(t,s) = t} \\ \dfrac{d\tilde{u}(t,s)}{dt} = 0, & \tilde{u}(0,s) = g(s) \Rightarrow \underline{\tilde{u}(t,s) = g(s)} \end{cases}$$

the char. are STRAIGHT LINES

the solution is constant along the charact.

Step 2 : we need to invert the function $(x(t,s), y(t,s))$

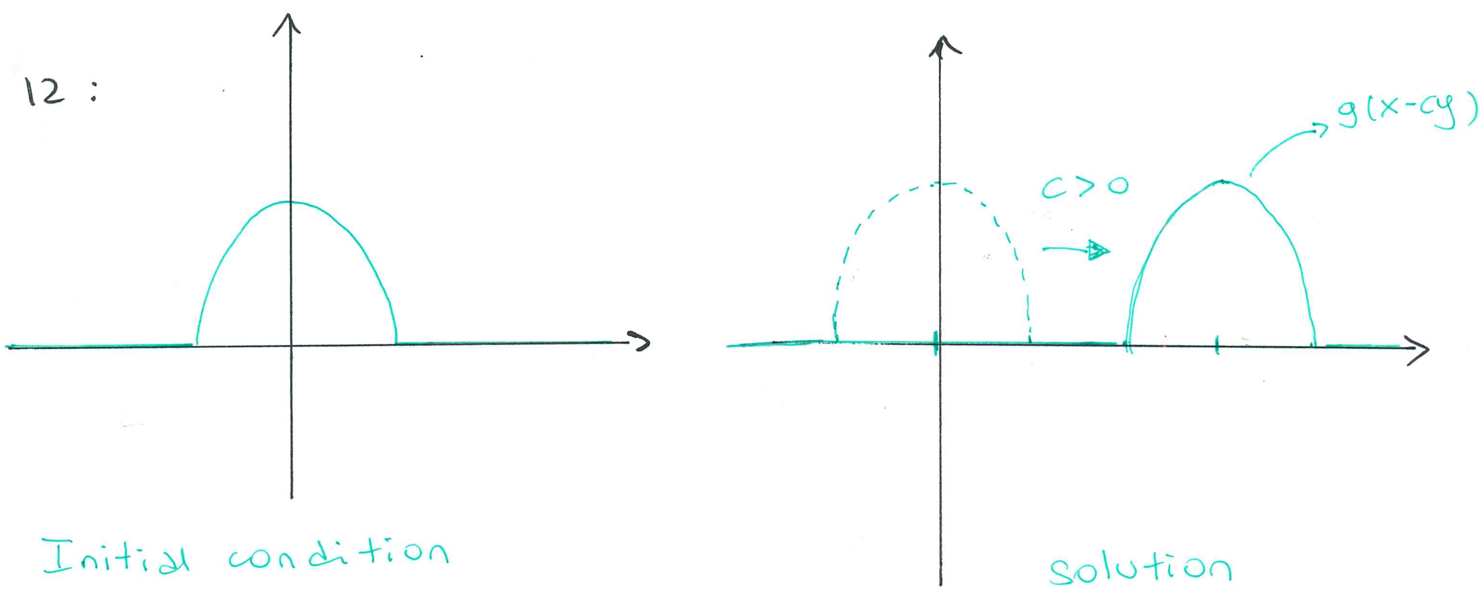
$$\begin{cases} y(t,s) = t & \Rightarrow \quad t(x,y) = y \\ x(t,s) = ct+s = cy+s & \Rightarrow \quad s(x,y) = x - cy \end{cases}$$

Step 3 : Finally

$$u(x,y) = \tilde{u}(t(x,y), s(x,y)) = g(s(x,y)) = g(x-cy)$$

$$\Rightarrow \underline{u(x,y) = g(x-cy)}$$ Travelling wave

Fig 12 :



Initial condition $g(x)$ $(y=0)$

solution $u(x,y) = g(x-cy)$ with $c>0$ after time $y>0$

Example (Burgers' equation (*)) → prototype of conservation law

$$(4) \begin{cases} u_y + u\,u_x = 0 \\ u(x,0) = h(x) \end{cases}$$

→ non-linear term, the velocity is NOT constant as in the transport equation

1st order, quasilinear

→ THE SPEED $c(u) = u$ DEPENDS ON THE SOLUTION ITSELF

This equation models the flow of a mass with concentration $u(x,y)$ where the speed of the flow depends on the concentation. The variable y has the physical interpretation of time, and $h(x)$ is the initial condition, so the concentration at time $y=0$.

Remark: (4) is in the form $u_y + c(u)u_x = 0$ with $c(u) = u$.

$u_y + u\,u_x = 0 \iff u_y + \left(\frac{1}{2}u^2\right)_x = 0$, thus the flux is $f(u) = \frac{1}{2}u^2$ while the wave speed is $c(u) = f'(u) = u$.

Since (4) is a first order equation we can use the M.o.C.

The parametrized initial condition is

$\Gamma(s) = (s, 0, h(s))$ and the characteristic eq. are given by:

the characteristics are STRAIGHT LINES

$$\begin{cases} \frac{d}{dt}x(t,s) = \tilde{u}(t,s), & x_0(s) = s & \Rightarrow & x(t,s) = s + h(s)y \\ \frac{d}{dt}y(t,s) = 1, & y_0(s) = 0 & \Rightarrow & y(t,s) = t \\ \frac{d}{dt}\tilde{u}(t,s) = 0, & \tilde{u}_0(s) = h(s) & \Rightarrow & \tilde{u}(t,s) = h(s) \end{cases}$$

(*) In the book is equation 2.41 and it is called Euler equation

Inverting as in the example of the transport eq. we obtain: $y = t$, $x = s + h(s)y$.

Therefore the solution of the PDE is implicitly given by

$$\tilde{u}(s + h(s)y, y) = h(s)$$

Note that the initial value of u (namely h) determines the slopes of the different characteristics.

Recalling that $x = s + h(s)y$ and $\tilde{u} = h(s)$ we have that $s = x - \tilde{u}y$, therefore the implicit solution can be written as:

$$(5) \quad \boxed{u(x,y) = h(x - \underline{u}y)}$$

the solution depends by the solution itself

Remark 1: (5) doesn't come unexpected (look back at the solution of the transport equation) and it is actually very general, indeed if you are solving

$$(6)\begin{cases} u_y + c(u)u_x = 0 & (x,y) \in \mathbb{R} \times (0, +\infty) \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R} \quad \text{(Initial Condition)} \end{cases}$$

Then u satisfies the implicit equation:

$$u(x,y) = u_0(x - c(u(x,y))y)$$

Implicit solution for a conservation law

Remark 2: We are studying the following problem:

$$\begin{cases} u_y + uu_x = 0, & x \in \mathbb{R}, y \in (0, +\infty) \\ u(x,0) = h(x) & x \in \mathbb{R} \end{cases}$$

The initial curve is $\Gamma = \mathbb{R} \times \{0\}$. Let us verify the transversality condition:

$$\det \begin{pmatrix} \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{pmatrix} = \det \begin{pmatrix} u & 1 \\ 1 & 0 \end{pmatrix} \neq 0$$

Therefore all the points of Γ are noncharacteristic and, if h is continuously differentiable theorem 2.10 ensures that the conservation law has a unique solution on some time interval $[0, y_c)$, where y_c is sufficiently small..

c stands for "critical"

Let us now determine the critical time y_c when the "classical" (or strong) solution breaks down.

In order to determine s, given (x,y), let us fix $y = \bar{y}$ and look at the map

$$s \longmapsto s + h(s)\bar{y} = x(\bar{y}, s).$$

Assume $h \in C^1$ we can compute

$$\frac{d}{ds}(s + h(s)\bar{y}) = 1 + h'(s)\bar{y}.$$

Assume also that, for all $s \in \mathbb{R}$, $1 + h'(s)\bar{y} > 0$.

This imply that the map $s \longmapsto x(\bar{y}, s)$ is strictly increasing, therefore there exists its unique inverse map. Thus, for $\bar{y}$ fixed, we can invert the relation

$$s \mapsto s + h(s)\bar{y}$$

provided that $1 + h'(s)\bar{y} > 0$.

If we assume for instance that h' is globally bounded, then $1 + h'(s)\bar{y} > 0$ for $\bar{y} \geq 0$ with $\bar{y}$ small enough.

Q: what is the first value $y > 0$ for which we cannot invert anymore the relation

$$s \mapsto s + h(s)\, y \;?$$

This is given by the first y for which there exists $s \in \mathbb{R}$ such that $1 + h'(s)\, y = 0$. If we call y_c such a y (again, c stands for "critical") we can say that

$$(7) \qquad y_c = \inf_{s \in \mathbb{R}\,:\, h'(s) < 0} \left\{ -\frac{1}{h'(s)} \right\}.$$

At the time y_c there is a problem with the solution u. To see it we can differentiate the relation $u(s + h(s)y, y) = h(s)$ w.r.t. s to get:

$$u_x(s + h(s)y, y)\,[1 + h'(s)y] = h'(s)$$

thus
$$u_x(s + h(s)y, y) = \frac{h'(s)}{1 + h'(s)y}$$

which shows that the derivative of u explodes when we take s such that $1 + h'(s)y_c = 0$. This shows that y_c is the critical time after which there is no smooth solution to the problem.

Remark 3 : Note that the formula (7) is specific to Burgers equation !

GENERAL FORMULA :

$$(*)\begin{cases} u_y + c(u)u_x = 0 , & (x,y) \in \mathbb{R} \times (0,+\infty) \\ u(x,0) = u_0(x) , & x \in \mathbb{R} \end{cases}$$

$c, u_0 \in C^1(\mathbb{R})$ and $c(u_0)$ bounded with bounded derivative. Then

$$\boxed{\begin{aligned} y_c &= \inf_{s \in \mathbb{R}\,:\, c(u_0(s))_s < 0} \left\{ \frac{-1}{c(u_0(s))_s} \right\} \\ &= \inf_{s \in \mathbb{R}\,:\, c(u_0(s))_s < 0} \left\{ \frac{-1}{c'(u_0(s)) \cdot u_0'(s)} \right\} \end{aligned}}$$

with the standard convention that $y_c = +\infty$ if $c(u_0(s))_s \geq 0$.

Then for $y_c > 0$ there exist a unique solution to $(*)$ in $[0, y_c)$, u satisfies the implicit equation

$$u(x,y) = u_0(x - c(u(x,y))y)$$

Remark 4 : (Characteristics for conservation laws)
Solutions of conservation laws are constant along their characteristics, which are STRAIGHT LINES: for each $s \in \mathbb{R}$, the characteristic through point $(x,y) = (s,0)$ is the line in the (x,y)-plane through $(s,0)$ with slope $\dfrac{1}{c(u_0(s))}$ and on this line u takes the constant value $u_0(s)$.

Geometric interpretation of y_c: crossing characteristics

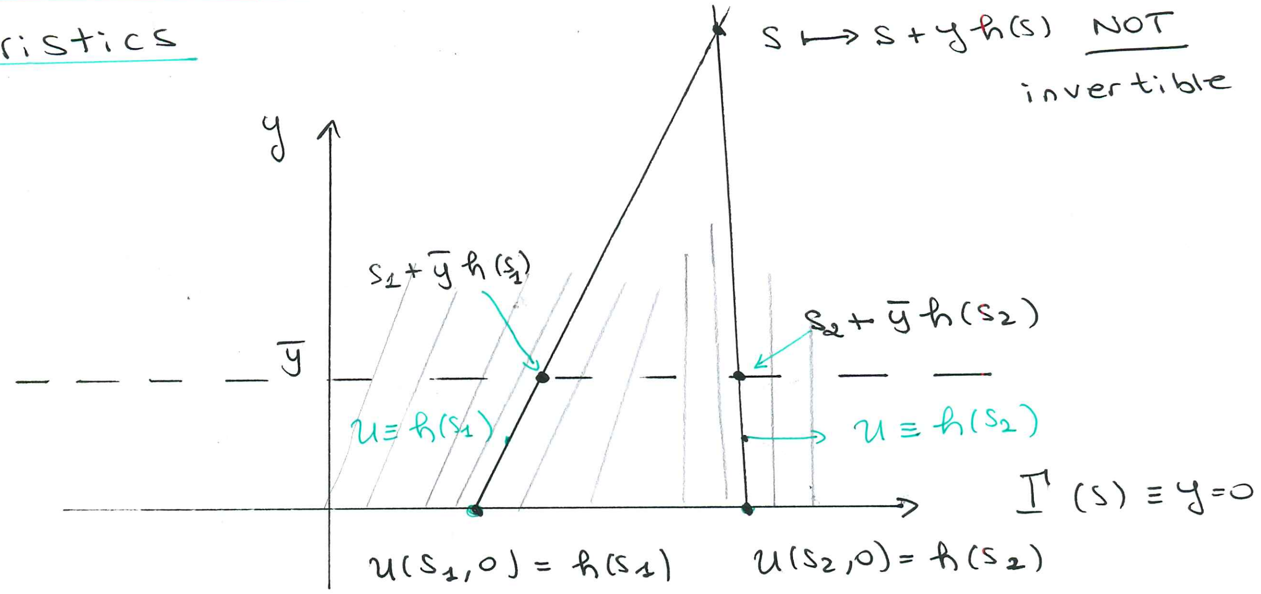


Fig 13:

If $c(u_0(s))_s < 0$ then there exists y_c time when the characteristics cross! Heuristically you can think about the latter condition as when a faster characteristics starts from a point behind a slower characteristics.

If $c(u_0(s))_s$ is never decreasing, there will not be any singularity, however such data are exceptional.

Notion of weak solutions

In the context of conservation laws we say that u is a classical or strong solution if u satisfies the classical formulation of conservation laws (1) and (2). To introduce a notion of weak solution we will use the so-called INTEGRAL FORMULATION of the conservation law:

$$(I) \quad \int_a^b u(x, y_2)\,dx - \int_a^b u(x, y_1)\,dx = -\int_{y_1}^{y_2} f(u(x,y)) \Big|_{x=a}^{x=b} dy$$

$\forall\ a, b,\ a < b,\ y_1 < y_2$ and $f(u)$ FLUX FUNCTION.

Now the classical formulation (1) (and (2)) makes no sense if u is not continuously differentiable but the integral formulation (I) still makes sense even if u is not continuous.

CLASSICAL FORMULATION $\Rightarrow$ INTEGRAL FORMULATION

INTEGRAL FORMULATION + REGULARITY OF u $\Rightarrow$ CLASSICAL FORMULATION

Let us suppose that the domain of definition of $u(x,t)$ is D, and that D is divided into sub-domains D_i, $i=1,..,n$. Assume that $u(x,y)$ is continuously differentiable in each D_i, $\forall i=1,..,n$.

Def: We define a weak solution $u(x,y)$ on $D = \bigcup_i D_i$ to be one satisfying the original PDE (1) (or (2)) in each D_i and the integral form (I) on D. The boundaries between regions D_i are curves $\Gamma = (\sigma(y), y)$ called shocks. *

Thus in our notion of weak solution we relax the requirement of a global classical solution and we allow solutions that are a combination of classical solutions on each D_i with possibly jumps between them.

We now will see a very important condition that has to be verified in order to have a global weak solution.

---

* in the book its indicated by $x = \gamma(y)$ while here I use $\sigma(y)$ and or $\gamma(y)$ but it's the same concept

RANKINE-HUGONIOT CONDITION

Let $x = \sigma(y)$ be a smooth curve in the (x,y) plane across which u is discontinuous. Assume that u, u_x, u_y have one sided limits as $x \to \sigma(y)^+$ and as $x \to \sigma(y)^-$. Choosing $a < \sigma(y), b > \sigma(y)$ the formula (I) becomes:

$$f(a,y) - f(b,y) = \frac{d}{dy}\int_a^{\sigma(y)} u(x,y)\,dx + \frac{d}{dy}\int_{\sigma(y)}^{b} u(x,y)\,dx$$

$$= \int_a^{\sigma(y)} u_y(x,y)\,dx + \sigma'(y)\,u(\sigma(y)^-, y) + \int_{\sigma(y)}^{b} u_y(x,y)\,dx$$

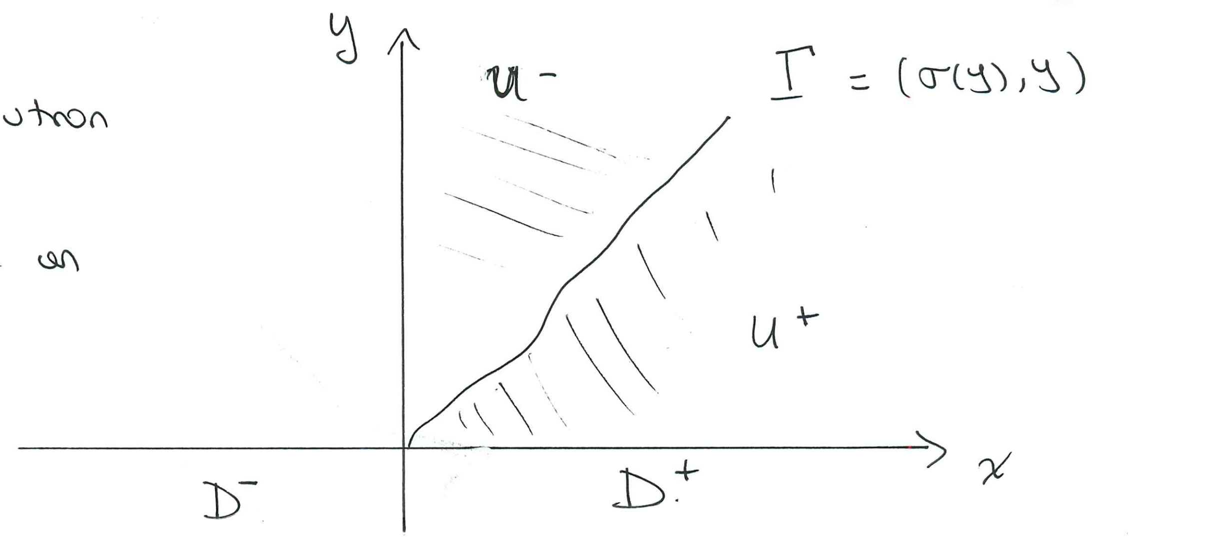
$$- \sigma'(y)\,u(\sigma(y)^+, y).$$

In the limit as $a \to \sigma(y)^-$ and $b \to \sigma(y)^+$ the integral vanish and we have:

$$f^+ - f^- = \sigma'(y)(u^+ - u^-) \Rightarrow \boxed{\sigma'(y) = \frac{f^+ - f^-}{u^+ - u^-}} \quad (RH)$$

The condition (RH) is called the Rankine-Hugoniot condition. Solutions that satisfy RH are called SHOCK WAVES

Fig 14: a solution with a jump discontinuity on a curve



Example (Breakdown time for Burgers' eq.)

$$\begin{cases} u_y + u u_x = 0 & \text{for } (x,y) \in \mathbb{R} \times (0,+\infty) \\ u(x,0) = e^{-x^2} & \text{for } x \in \mathbb{R} \end{cases}$$

Here the velocity is $c(u) = u$, $u_0(x) = e^{-x^2}$.

Observe that $c(u_0(s)) = e^{-s^2}$ is decreasing for $s>0$.

This conservation law has a classical solution for $y \in [0, y_c)$ where

$$y_c = \min_{s>0} \left\{ \frac{-1}{c'(u_0(s))\, u_0'(s)} \right\} = \min_{s>0} \frac{e^{s^2}}{2s}$$

$$c'(u_0(s)) \cdot u_0'(s) = -2s\, e^{-s^2}$$

Let now $y_*(s) := \dfrac{e^{s^2}}{2s}$. $\lim\limits_{s \to 0} y_*(s) = \lim\limits_{s \to +\infty} y_*(s) = +\infty$

therefore y_* is minimised when

$$\frac{dy_*}{ds} = 0 \iff e^{s^2}\left(1 - \frac{1}{2s^2}\right) = 0 \iff s^2 = \frac{1}{2}$$

Therefore $s = \dfrac{1}{\sqrt{2}}$ is the unique critical point of y_* in $(0,\infty)$. We conclude that $y_c = y_*\left(\frac{1}{\sqrt{2}}\right) = \dfrac{e^{1/2}}{\sqrt{2}}$

For $y < y_c$ the solution is:

$$u(x,y) = u_0(s) = e^{-s^2}$$

where s is the unique solution of $x = s + c(u_0(s))y$

$$\iff x = s + e^{-s^2} y$$