Recall from Lecture 4.

What are conservation laws? They are a class of 1st order PDEs describing the evolution of a conserved quantity.

Classical form of a conservation law: $u = u(x,y)$, y time variable:

$$(1) \quad u_y + \underbrace{f(u)_x}_{f'(u)u_x} = 0 \quad , \quad f(u) \text{ FLUX}, \ c(u) \text{ SPEED}.$$

$$(2) \quad u_y + c(u)u_x = 0 \quad , \quad c(u) = f'(u)$$

We can apply the M.o.C. to solve conservation laws. (they are 1st order, quasilinear eq). If we have

$$\begin{cases} u_y + c(u)u_x = 0 \\ u(x,0) = u_0(x) \end{cases} , \quad c(u), u_0 \text{ continuous}$$

$\Gamma(s) = (s, 0, u_0(s))$

Transversality condition: $\det \begin{pmatrix} c(u) & 1 \\ 1 & 0 \end{pmatrix} = -1 \neq 0 \quad \forall s$

Thus we can apply thm 2.10 that guarantees local existence and uniqueness on some time interval $[0, y_c)$.

Formula for $y_c := \inf\limits_{c(u_0(s))_s < 0} \left\{ \dfrac{-1}{c'(u_0(s))\, u_0'(s)} \right\}$

u solves the implicit equation:

$$u(x,y) = u_0(x - c(u(x,y))y) \quad , \quad (x,y) \in \mathbb{R} \times [0, y_c)$$

What happens after time y_c? We can still have solutions but we need to relax our definition of solution. We introduce the notion of weak solution. To do so, we define the INTEGRAL FORMULATION of a conservation law:

Recall of Lecture 4 :

$$(I) \quad \int_a^b u(x,y_2)\,dx - \int_a^b u(x,y_1)\,dx = -\int_{y_1}^{y_2} f(u(x,y))\Big|_{x=a}^{x=b} dy$$

$\forall\, a<b$, $u_1 < u_2$ and $f(u)$ FLUX FUNCTION.

Weak solution of (1) (or (2)): Let $D = \bigcup_{i=1}^{n} D_i$ be the domain of definition of u. Let $u(x,y)$ be continuously differentiable in each D_i, $i=1,\dots,n$. Then $u(x,y)$ is a weak solution of (1) (or (2)) if $u(x,y)$ satisfies the original PDE ((1) or (2)) in each D_i, and it satisfies (I) in the whole domain D. The boundaries between regions D_i are curves $\Gamma = (\sigma(y), y)$ called shocks. Moreover the shocks have to satisfy the so called Rankine - Hugoniot condition:

$$\boxed{\sigma'(y) = \frac{f^+ - f^-}{u^+ - u^-} \qquad (RH).}$$

Such weak solutions are also said to be shock waves.

Good news : now that we have the notion of weak solution we don't fear discontinuities (shocks) and we can use this notion to model several situations of interest (traffic jams, sonic booms ...)

Bad news : nothing is for free. Once we relax our notion of solution we lose the uniqueness (guaranteed by thm 2.10). Therefore we will have several solutions to the same problem.

In today's lecture we will learn how to select the "best" weak solution to our problem.

Remark : $u_y + u u_x = 0 \iff u_y + \left(\frac{u^2}{2}\right)_x = 0$

(Burgers' eq.), $f(u) = \frac{u^2}{2}$ flux for Burgers' eq.

(Section 2.7, 3.1, 3.2)

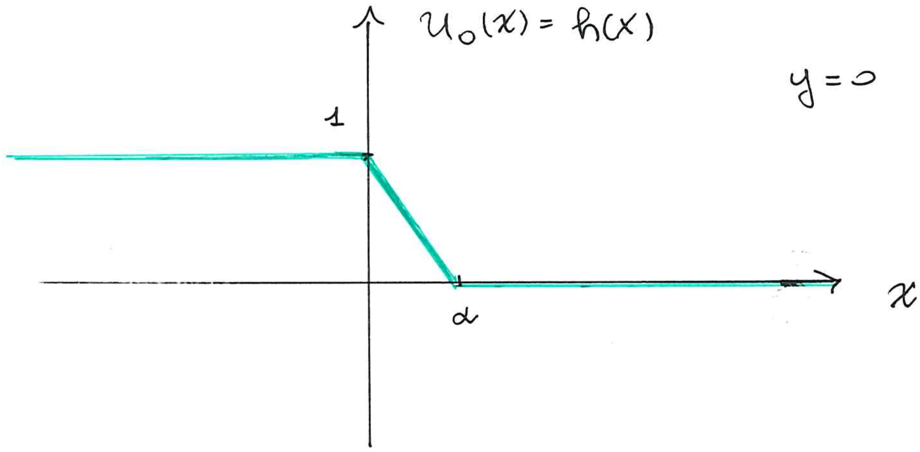
Conservation laws : the entropy condition.

Let now see some examples:

Example 2.14 (PR): Consider the problem

$$\begin{cases} u_y + u u_x = 0 \rightsquigarrow \text{Burgers' equation}, \quad c(u) = u, \quad f(u) = \frac{u^2}{2} \\ u(x,0) = h(x) = \begin{cases} 1 & \text{for } x \le 0 \\ 1 - \frac{x}{\alpha} & \text{for } 0 \le x \le \alpha \\ 0 & \text{for } x \ge \alpha \end{cases} \end{cases}$$

Initial condition:



Since the initial condition $h(x)$ is not monotone increasing the solution will develop a singularity at time $y = y_c$.

$$y_c = \inf_{c(u_0(s))_s < 0} \left\{ \frac{-1}{c'(u_0(s))\, u_0'(s)} \right\}, \quad c(u_0(s)) = c(h(s)) = h(s)$$

The initial datum $h(x)$ is not differentiable, nevertheless for $s \in (0, \alpha)$, $c(h_0(s)) = 1 - \frac{s}{\alpha}$ is decreasing and we expect u to become discontinuous at time

$$y_c = \inf_{s \in (0,\alpha)} \left\{ - \frac{1}{c(h(s))_s} \right\} = \inf_{s \in (0,\alpha)} \left\{ + \alpha \right\} = \alpha .$$

M.o.C.
$$\begin{cases} x_t = a = u & \Rightarrow x(t,s) = s + t h(s) \\ y_t = b = 1 & \Rightarrow y(t,s) = t \\ \tilde{u}_t = 0 & \Rightarrow \tilde{u}(t,s) = h(s) \end{cases}$$

We have that, for $y < y_c = \alpha$ the solution is $u(x,y) = h(s)$, where (x,y) lies on the characteristic through $(s,0)$. Since h is defined piecewise we have to consider 3 cases:

a) If $s \leq 0$, then $h(s) \equiv 1$ and the characteristics have equation $x = s + c(h(s))y = s + y$. $(\Rightarrow s = x - y)$

Therefore $u(x,y) = h(s) = 1$ for $s \leq 0 \iff x \leq y$.

The characteristic lines $y = x - s$ are shown in blue in the following figure;

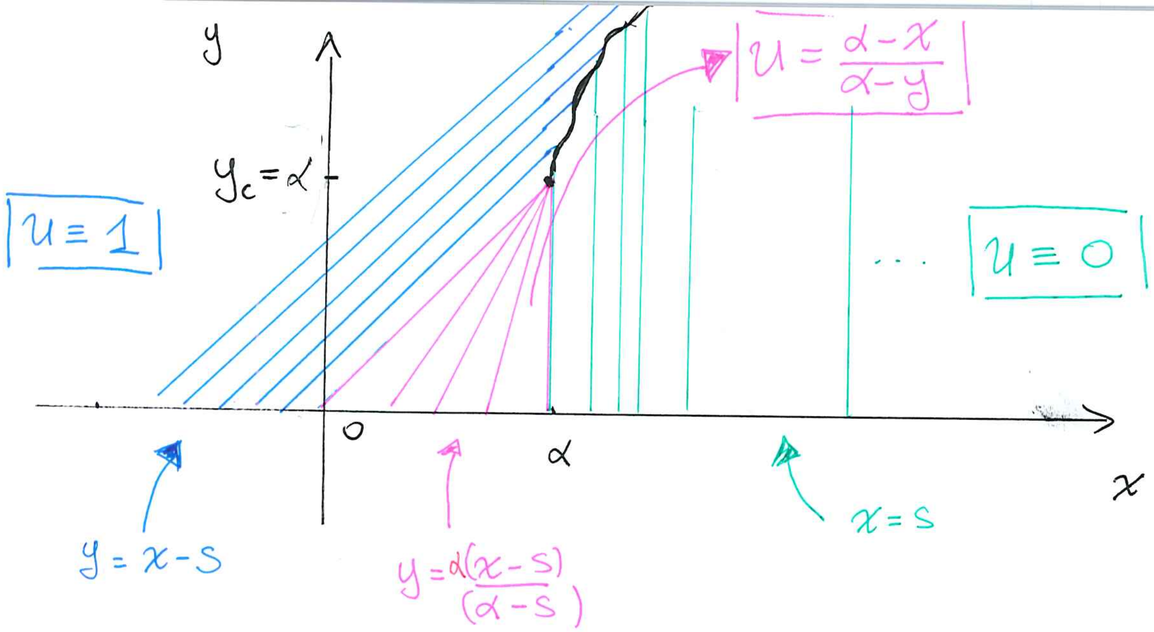
b) If $s \geq \alpha$, then $h(s) \equiv 0$ and the characteristics have equation $x = s + c(h(s))y = s$. Therefore $u(x,y) = h(s) = 0$ for $s \geq \alpha \iff x \geq \alpha$. The characteristic lines $x = s$ are in green.

c) Finally, for $0 \leq s \leq \alpha$, then $h(s) = 1 - \frac{s}{\alpha}$ and the characteristics have equation $x = s + c(h(s))y =$
$= s + \left(1 - \frac{s}{\alpha}\right) y \iff s = \frac{\alpha(x-y)}{\alpha - y}$. Therefore

$$u(x,y) = h(s) = 1 - \frac{s}{\alpha} = 1 - \frac{1}{\alpha}\left(\frac{\alpha(x-y)}{\alpha-y}\right) = 1 - \left(\frac{x-y}{\alpha-y}\right) = \frac{\alpha - x}{\alpha - y}$$

if $0 \leq s \leq \alpha \iff y \leq x \leq \alpha$ $\quad (s(\alpha - y) = \alpha x - \alpha y \Rightarrow$
$\alpha x = s\alpha - sy + \alpha y$ and substitute the inequality $0 \leq s \leq \alpha)$

The characteristic lines $y = \frac{\alpha(x-s)}{(\alpha - s)}$ are shown in pink.



At time $y_c = \alpha$ characteristics intersect.
From $y = y_c = \alpha$ a shock (the black curve) will appear in the graph of u, jumping from 1 to zero.
To find the shock curve we impose the Rankine-Hugoniot condition, so γ (or σ) satisfies

$$\gamma'(y) = \frac{f(u^+) - f(u^-)}{u^+ - u^-} = \frac{1}{2} \frac{(u^+)^2 - (u^-)^2}{u^+ - u^-} = \frac{1}{2} \frac{(1-0)}{(1-0)} = \frac{1}{2}$$

Since $\gamma'(y) = \frac{1}{2}$, we deduce that γ is a linear function and, by the condition that the shock starts at the point (α, α) we get:

$$\gamma(y) = \alpha + \frac{1}{2}(y - \alpha), \; y \geq \alpha . \qquad \left(x = \frac{1}{2} y + \frac{\alpha}{2} \right)$$

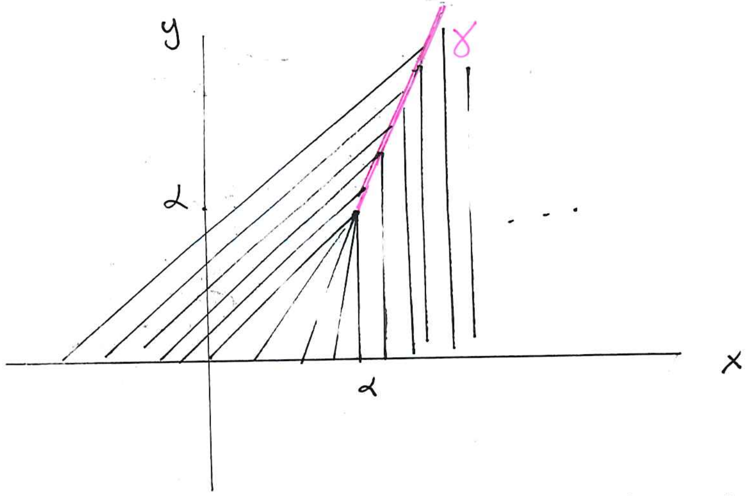
Therefore, for $y \geq \alpha$ we can write the solution for u as

$$u(x,y) = \begin{cases} 1 & \text{for } x < \gamma(y) \\ 0 & \text{for } x > \gamma(y) \end{cases}$$

Note that this is a weak solution (or shock wave).
In conclusion we constructed the following weak solution

$$u(x,y) = \begin{cases} 1 & \text{if } x \leq y, \; y \in [0, \alpha) \\ \frac{\alpha - x}{\alpha - y} & \text{if } y \leq x \leq \alpha, \; y \in [0, \alpha) \\ 0 & \text{if } x \geq \alpha, \; y \in [0, \alpha) \\ 1 & \text{if } x < (y + \alpha)/2, \; y \in [\alpha, +\infty) \\ 0 & \text{if } x > (y+\alpha)/2, \; y \in [\alpha, +\infty) \end{cases}$$

Characteristics and shock line (in pink) for example 2.14:



In the next example we see how, by allowing for weak solutions we lose uniqueness:

Example 2.15: Consider now the problem

$$\begin{cases} u_y + u u_x = 0 & \leadsto \text{Burgers' equation} \\ u(x,0) = h(x) = \begin{cases} 0 & \text{for } x \le 0 \\ \frac{x}{\alpha} & \text{for } 0 \le x \le \alpha \\ 1 & \text{for } x \ge \alpha \end{cases} \end{cases}$$

Since this time $c(h(s))_s = h'(s) \geq 0$ there is no critical time $y_c > 0$ where the characteristics intersect.
On the contrary, the characteristic diverge: this situation is called an expansion wave.
The characteristic of x is given by:

$$x(t,s) = s + y h(s) \underset{(*)}{=} \begin{cases} s & \text{for } s \le 0 \\ s + y \cdot \frac{s}{\alpha} & \text{for } s \in [0,\alpha] \\ s + y & \text{for } s \ge \alpha \end{cases}$$

Thus, inverting each relation between s and x:

$$s = \begin{cases} x & \text{for } \{s \le 0\} \overset{(*)}{=} \{x \le 0\} \\ \frac{\alpha x}{\alpha + y} & \text{for } \{s \in [0,\alpha]\} = \{0 \le \frac{\alpha x}{\alpha+y} \le \alpha\} \overset{(*)}{=} \{0 \le x \le y + \alpha\} \\ x - y & \text{for } \{s \ge \alpha\} = \{x - y \ge \alpha\} \overset{(*)}{=} \{x \ge y + \alpha\} \end{cases}$$

Since $u(x,y) = h(s)$, this yields

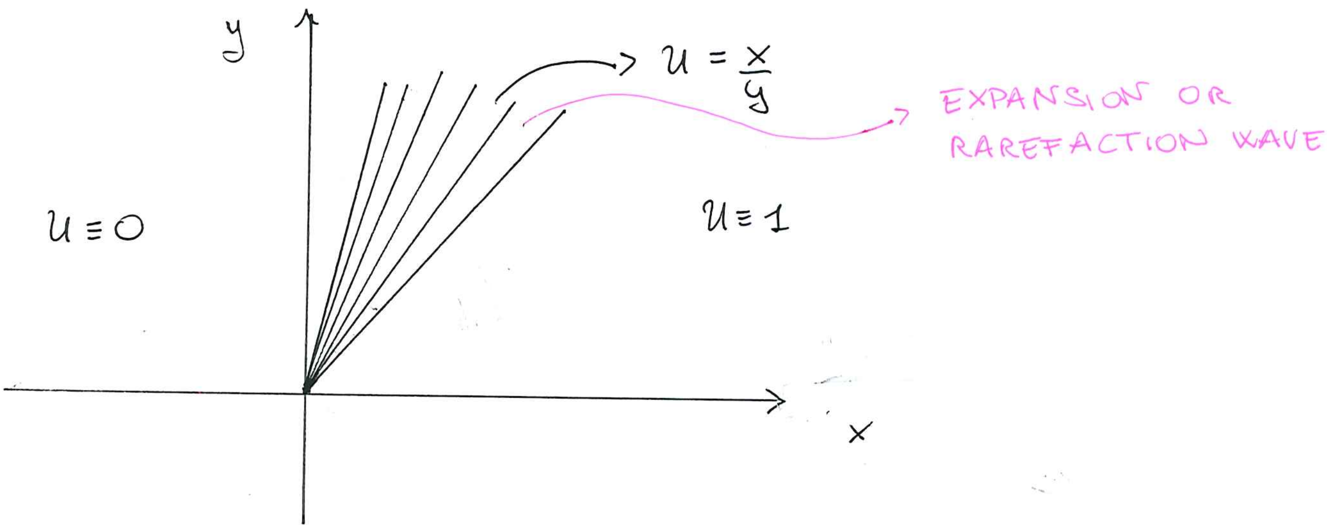
$$u(x,y) = \begin{cases} 0 & \text{for } x \le 0 \\ \frac{s}{\alpha} = \frac{x}{\alpha + y} & \text{for } 0 \le x \le y+\alpha \\ 1 & \text{for } x \ge y+\alpha \end{cases} \qquad \text{(S)} \quad \text{(classical solution)}$$

Let us now look at the case when $\alpha \to 0$. Then $h(x)$ becomes the step function

$$h(x) = \begin{cases} 0 & \text{for } x \le 0 \\ 1 & \text{for } x \ge 1 \end{cases}$$

By taking the solution (S) above and letting $\alpha \to 0$ we obtain:

$$u(x,y) = \begin{cases} 0 & \text{for } x \le 0 \\ \frac{x}{y} & \text{for } 0 < x < y \\ 1 & \text{for } x \ge y \end{cases} \quad \leftarrow \text{this is a classical solution}$$



In principle we can still find a solution with a shock, because $u \equiv 0$ for $x < 0$ and $u \equiv 1$ for $x > y$. A shock should be a curve $(\gamma(y), y)$ satisfying $\gamma(0) = 0$ and, by (RH) $\gamma'(y) = \frac{u^+ + u^-}{2} = \frac{1}{2}$, which leads to $\gamma(y) = \frac{y}{2}$.

Therefore another solution would be :

$$u(x,y) = \begin{cases} 0 & \text{for } x \le \frac{y}{2} \\ 1 & \text{for } x > \frac{y}{2} \end{cases}$$

If a conservation law does not have a unique weak solution, then how can we select the "right" one ?

Definition: Entropy condition. A weak solution satisfy the entropy condition if characteristics only enter shocks but not emanate from them.

This is expressed through the condition: $x = \gamma(y)$ shock satisfy the entropy condition if

$$c(u^+) < \gamma' < c(u^-)$$

equivalently $\quad f'(u^+) < \gamma' < f'(u^-) \quad$ where f is the flux and $f'(u) = c(u)$.

One can see that the solution

$$u(x,y) = \begin{cases} 0 & \text{for } x \le 0 \\ \frac{x}{y} & \text{for } 0 \le x \le y \\ 1 & \text{for } x > y \end{cases}$$

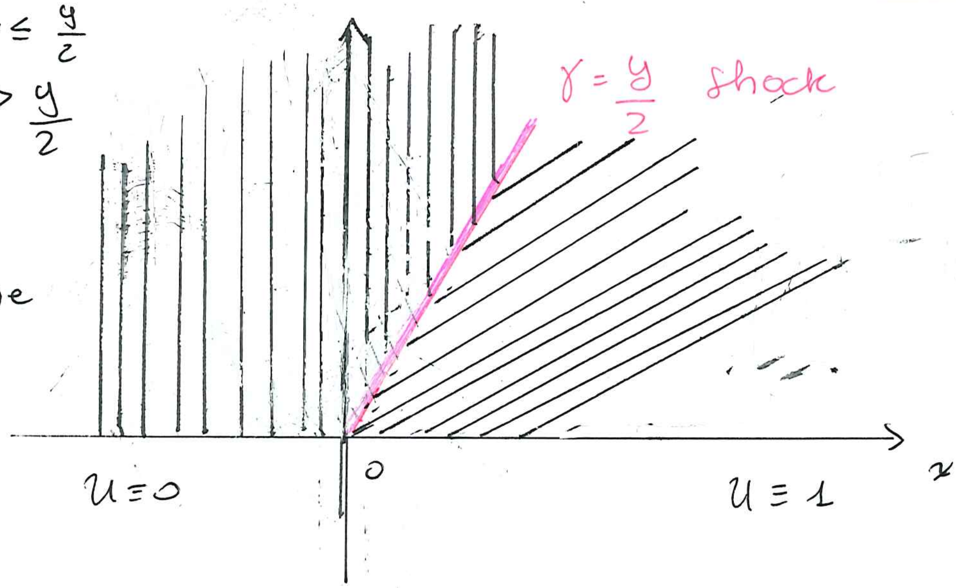
satisfy the entropy condition trivially because it has no shocks, while

$$u(x,y) = \begin{cases} 0 & \text{for } x \le \frac{y}{2} \\ 1 & \text{for } x > y/2 \end{cases}$$

$c(u) = u, \quad \gamma' = \frac{1}{2} \quad \Rightarrow \quad \underbrace{1}_{c(u^+)} < \frac{1}{2} < \underbrace{0}_{c(u^-)}$ false $\Rightarrow$ do not satisfy the Entropy condition.

$$u(x,y) = \begin{cases} 0 & \text{for } x \le \frac{y}{2} \\ 1 & \text{for } x > \frac{y}{2} \end{cases}$$

Characteristics emerging from the shock.



Classification of linear second order PDEs

Assuming $u_{xy} = u_{yx}$ (this is always the case for us) the general linear second order PDE in two independent variables has the form:

$$L[u] = \underbrace{a u_{xx} + 2b u_{xy} + c u_{yy}}_{\substack{\text{Second order terms} \\ \text{are the leading terms}}} + \underbrace{d u_x + e u_y + f u}_{\substack{\text{lower order} \\ \text{terms}}} = g$$

$a u_{xx} + 2b u_{xy} + c u_{yy}$ is also said principal part because the behaviour of the PDE is determined by a, b and c.

Remark : $a = a(x,y)$, $b = b(x,y)$, $c = c(x,y)$ because we are studying LINEAR second order PDEs.

As we already said, being able to properly classify the PDE we wish to investigate allows us to choose the correct method (if it exists!) to tackle the PDE. Knowing the "type" of the equation allows one to use the relevant methods to solve it, which can be quite different depending on the type of the equation.

You probably encountered conic sections and quadratic forms which are usually classified into parabolic, elliptic and hyperbolic based on the discriminant $b^2 - 4ac$, the same can be done for a second order PDE at a given point.

Classification: Given a point (x_0, y_0) consider the value $\delta(L)(x_0, y_0) = b^2(x_0, y_0) - a(x_0, y_0)c(x_0, y_0)$. At the point (x_0, y_0) the PDE is said to be:

- hyperbolic if $\delta(L)(x_0, y_0) > 0$
- parabolic if $\delta(L)(x_0, y_0) = 0$
- elliptic if $\delta(L)(x_0, y_0) < 0$

Remark: Since there is the convention that the xy term is $2b$ then the discriminant becomes

$$(2b)^2 - 4ac = 4(b^2 - ac)$$ and the 4 can be dropped.

Remark: This classification is a local property. However we will often study PDEs with constant coefficients where the classification is global.

Example: Consider the following PDEs:

- The wave equation $u_{tt} - u_{xx} = 0$ is hyperbolic (we use variables (x, t) instead of (x, y))
- The heat equation $u_t - u_{xx} = 0$ is parabolic
- The Laplace equation $u_{xx} + u_{yy} = 0$ is elliptic (here we use (x, y) intended as spatial variables)

Similarly to what happens with second order algebraic equations, we can use a nondegenerate change of variables to reduce the equation to a simpler form

Definition - Change of coordinates. A transformation $(x,y) \mapsto (\xi,\eta) = (\xi(x,y), \eta(x,y))$ is called a change of coordinates near a point (x_0, y_0) if

$$\det \begin{pmatrix} \partial_x \xi & \partial_y \xi \\ \partial_x \eta & \partial_y \eta \end{pmatrix} \Bigg|_{(x,y)=(x_0,y_0)} \neq 0$$

Fact: 2nd order PDEs can be transformed in the so called canonical form by using a change of coordinates $u(x,y) \longmapsto w(\xi,\eta) = w(\xi(x,y), \eta(x,y))$.

These are the canonical forms:

- Hyperbolic: $\boxed{w_{\xi\eta}} + \tilde{d} w_\xi + \tilde{e} w_\eta + \tilde{f} w = \tilde{g}$
- Parabolic: $\boxed{w_{\xi\xi}} + \tilde{d} w_\xi + \tilde{e} w_\eta + \tilde{f} w = \tilde{g}$
- Elliptic: $\boxed{w_{\xi\xi} + w_{\eta\eta}} + \tilde{d} w_\xi + \tilde{e} w_\eta + \tilde{f} w = \tilde{g}$

Example: Consider the wave eq. $u_{tt} - c^2 u_{xx} = 0$, $t \geq 0$.

Let us use the transformation

$$\begin{cases} \xi = x + ct \\ \eta = x - ct \end{cases}$$

* we will see this computation next week.

This gives us $u(x,t) = w(\xi,\eta) = w(x+ct, x-ct)$. Plugging this into the wave eq. gives:*

$u_{tt} - c^2 u_{xx} = -4c^2 w_{\xi\eta} = 0$. By dividing by $-4c^2$ we get

$w_{\xi\eta} = 0$ which is the canonical form for hyperbolic eq.