- In this lecture we will study the 1D wave equation (the archetype of hyperbolic equation) on the real line
- We will use the reduction to the canonical form to show that the general solution of the 1D wave equation can be decomposed as superposition of a forward and a backward travelling wave.
- We will introduce the D'Alembert formula that gives us an explicit solution of the Cauchy problem.

Section 4.1, 4.2, 4.3

We will consider the homogeneous wave equation in one spatial variable on the real line.
Usually real life applications of the wave eq. take place on finite intervals. In that case we would need to deal with boundary conditions but for now we consider the simplified setting of the absence of boundary conditions in order to make some general considerations.

Canonical form and general solution

The homogeneous one dimensional wave equation is a hyperbolic second-order differential equation of the form:

$$u_{tt} - c^2 u_{xx} = 0 \quad , \quad (x,t) \in \mathbb{R} \times (0,+\infty)$$

where $c \in \mathbb{R}$ denotes the wave speed.

- Why wave equation? It describes, well, waves. Wave propagation appears in a huge pletora of different physical situations: water wave propagation, sound waves, seismic waves and light waves. It arises in acoustic, electromagnetism, and fluid dynamics.

Given $u_{tt} - c^2 u_{xx} = 0$ (1), we introduce the new variables:

$$\zeta(x,t) = x + ct \quad \text{and} \quad \eta(x,t) = x - ct .$$

Set $w(\zeta,\eta) = u(x(\zeta,\eta), t(\zeta,\eta))$. Using the chain rule for the function $u(x,t) = w(\zeta(x,t), \eta(x,t))$ we obtain:

$$u_t = \frac{\partial}{\partial t} w(\zeta(x,t), \eta(x,t)) = w_\zeta(\zeta(x,t), \eta(x,t)) \cdot \zeta_t(x,t)$$

$$+ w_\eta(\zeta(x,t), \eta(x,t)) \cdot \eta_t(x,t) = w_\zeta \zeta_t + w_\eta \eta_t$$

$$u_x = \frac{\partial}{\partial x} w(\zeta(x,t), \eta(x,t)) = w_\zeta(\zeta(x,t), \eta(x,t)) \cdot \zeta_x(x,t) +$$

$$+ w_\eta(\zeta(x,t), \eta(x,t)) \eta_x(x,t) = w_\zeta \zeta_x + w_\eta \eta_x$$

Since $\zeta_x = \eta_x = 1$ and $\zeta_t = c$, $\eta_t = -c$ we have:

$$u_x(x,t) = w_\zeta(\underbrace{x+ct}_{\zeta}, \underbrace{x-ct}_{\eta}) + w_\eta(x+ct, x-ct)$$

$$u_t(x,t) = c\left[w_\zeta(x+ct, x-ct) - w_\eta(x+ct, x-ct)\right]$$

- Differentiating again w.r.t. x ant t the above expressions we get:

$$u_{tt} = \frac{\partial}{\partial t}\left[c\left(w_\zeta - w_\eta\right)\right] = c^2\left(w_{\zeta\zeta} - 2w_{\zeta\eta} + w_{\eta\eta}\right)$$

$$u_{xx} = \frac{\partial}{\partial x}\left[w_\zeta + w_\eta\right] = w_{\zeta\zeta} + 2w_{\zeta\eta} + w_{\eta\eta}$$

Plugging in the wave eq. we obtain:

$$0 = u_{tt} - c^2 u_{xx} = c^2 \left[\cancel{w_{\xi\xi}} - 2w_{\xi\eta} + \cancel{w_{\eta\eta}} - 2w_{\xi\eta} - \cancel{w_{\eta\eta}} - \cancel{w_{\xi\xi}} \right]$$

$$= -4c^2 w_{\xi\eta}.$$

Thus we have $w_{\xi\eta} = 0$ ← CANONICAL FORM FOR THE WAVE EQ.

Notice that $w_{\xi\eta} = \frac{\partial}{\partial\eta}(w_\xi) = 0$. This implies that w_ξ is independent of η, therefore we can write it as $w_\xi(\xi,\eta) = f(\xi)$, for some function $f : \mathbb{R} \to \mathbb{R}$.

Then we integrate:

$$w(\xi,\eta) = \int_0^\xi f(\alpha)\,d\alpha + G(\eta) \quad \text{with} \quad G(\eta) = w(0,\eta).$$

If we call $F(\xi) = \int_0^\xi f(\alpha)\,d\alpha$ we can write the general solution for the equation $w_{\xi\eta} = 0$ as follows:

$$w(\xi,\eta) = F(\xi) + G(\eta)$$

$\left(F, G \in C^2(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} : f' \text{ and } f'' \text{ exist and are continuous} \} \right)$

Thus, in the original variables, the general solution of the 1D wave equation is

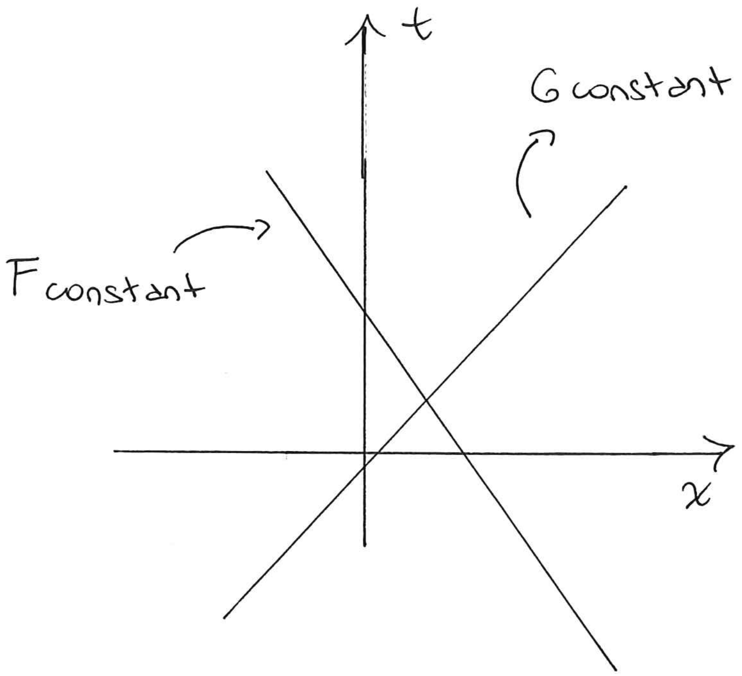
$$(2) \quad u(x,t) = \underbrace{F(x+ct)}_{\text{BACKWARD WAVE}} + \overbrace{G(x-ct)}^{\text{FORWARD WAVE}}.$$

If u solves (1) then there exist $F, G \in C^2(\mathbb{R})$ such that (2) holds. Conversely any two functions $F, G \in C^2(\mathbb{R})$ give a solution of (1) via the formula (2).

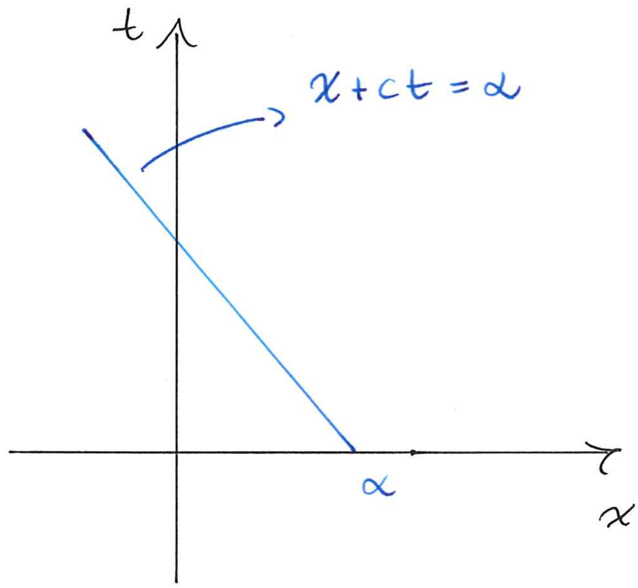
- $G(x-ct)$ represents a wave moving to the right with velocity $c>0$ ⇝ forward wave
- $F(x+ct)$ is a wave moving to the left with velocity $c>0$ ⇝ backward wave.

Remark: $F(x+ct)$ and $G(x-ct)$ are constant along lines of the form $x+ct=\alpha \in \mathbb{R}$ and $x-ct=\beta \in \mathbb{R}$ respectively. Those lines are called characteristics.

(2) shows that any solution of the 1D wave eq. is the sum of two traveling waves.



The characteristics on which F and G are constant



The backward wave $F(x+ct)$

Remark: for the wave equation the characteristics are straight lines in the (x,t) plane with slopes $\pm 1/c$.

As for first order PDEs, the "information" is propagated along these curves.

The equation (2) is valid for $F, G \in C^2(\mathbb{R})$.
Let now extend the validity of (2).

Consider F, G real piecewise continuous functions. Let's approximate F and G by two sequences of C^2 functions $\{F_n\}$, $\{G_n\}$. Hence we demand that:

1. $F_n, G_n \in C^2$ for all $n \in \mathbb{N}$;
2. $F_n \longrightarrow F$ at all continuity points of F;
3. $G_n \longrightarrow G$ at all continuity points of G.

Then the function

$$u_n(x,t) = F_n(x+ct) + G_n(x-ct)$$

is a solution of (1) in the classical sense. Sending n to infinity we obtain the limiting function

$$u(x,t) = F(x+ct) + G(x-ct)$$

that is not necessarily smooth enough to be called a "classical" or "strong" solution, but we call a function $u(x,t)$ that satisfies (2) with piecewise continuous functions F, G, a GENERALIZED SOLUTION of the wave equation.

Remark: Assume that u is a smooth function except at (x_0, t_0). Either F is not smooth at $x_0 + ct_0$ and/or G is not smooth at $x_0 - ct_0$. Then there are two characteristics passing through (x_0, t_0):

$$x - ct = x_0 - ct_0 \quad , \quad x + ct = x_0 + ct_0 .$$

Thus for any time $t_1 \neq t_0$ u is smooth except at one or two points

$x_\pm$ that satisfy:

$$x^- - ct_1 = x_0 - ct_0 \quad , \quad x^+ + ct_1 = x_0 + ct_0 .$$

The singularities of solutions of the wave equation are traveling only along characteristics.

↪ this is a typical feature of hyperbolic eq.

The Cauchy problem and d'Alembert's formula

The Cauchy problem for the homogeneous 1D wave equation is given by

$$\text{PDE} \rightarrow \begin{cases} u_{tt} - c^2 u_{xx} = 0 , & (x,t) \in \mathbb{R} \times (0,+\infty) \\ u(x,0) = f(x) & \rightsquigarrow f \text{ is the AMPLITUDE at } t=0 \\ u_t(x,0) = g(x) & \rightsquigarrow g \text{ represents the VELOCITY at } t=0 \end{cases} \quad (*)$$

I.C. (initial conditions: the last two lines)

→ A solution to the above Cauchy problem can be thought as the amplitude of the vibration of an infinite string.

A classical solution for the C.P. is a function u that is twice continuously differentiable for all $t \in \mathbb{R}^+$, u and u_t continuous in $[0,+\infty)$, and solving $(*)$.

Since the general solution to (1) is given by (2) we need to find F and G using the initial conditions.

By $u(x,0) = f(x)$ we deduce:

$$f(x) = u(x,0) = F(x) + G(x)$$

By $u_t(x,0) = g(x)$ we get:

$$g(x) = u_t(x,0) = \frac{d}{dt}\left[F(x+ct) + G(x-ct)\right]\Big|_{t=0} = c\left[F'(x) - G'(x)\right]$$

From

$$g(x) = c\,(F' - G')(x)$$

we obtain

$$(F-G)'(x) = \frac{1}{c}\,g(x) \Rightarrow F(x) - G(x) = \frac{1}{c}\int_0^x g(y)\,dy + [F(0) - G(0)]$$

Hence we have a system :

$$\begin{cases} F(x) + G(x) = f(x) \\ F(x) - G(x) = \dfrac{1}{c}\displaystyle\int_0^x g(y)\,dy + (F(0) - G(0)) \end{cases}$$

Adding the equations we get :

$$2F(x) = f(x) + \frac{1}{c}\int_0^x g(y)\,dy + (F(0) - G(0)).$$

Subtracting the second eq. from the first :

$$2G(x) = f(x) - \frac{1}{c}\int_0^x g(y)\,dy + (F(0) - G(0))$$

Therefore the solution of (*) is given by :

$$u(x,y) = F(x+ct) + G(x-ct) = \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(y)\,dy +$$

$$+ \cancel{\frac{F(0) - G(0)}{2}} + \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_0^{x-ct} g(y)\,dy - \cancel{\frac{F(0) - G(0)}{2}}$$

$$\Rightarrow \boxed{u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy.} \quad \checkmark$$

d'Alembert's formula

Remark : the value of the solution at (x,t) is ONLY INFLUENCED by the values in $[x-ct, x+ct]$.

Example 4.2.

Consider the Cauchy problem (*) with given initial conditions and $c = 1$:

$$f(x) = \begin{cases} 0 & \text{for } |x| > 1 \\ x+1 & \text{for } -1 \le x \le 0 \\ 1-x & \text{for } 0 \le x \le 1 \end{cases} \qquad u(x,0) = f(x)$$

$$g(x) = \begin{cases} 0 & \text{for } |x| > 1 \\ 1 & \text{for } -1 \le x \le 1 \end{cases} \qquad u_t(x,0) = g(x)$$

a) Evaluate u at the point $(1, \frac{1}{2})$. Let's use d'Alembert's formula:

$$u(1, \tfrac{1}{2}) = \frac{f(1+\frac{1}{2}) + f(1-\frac{1}{2})}{2} + \frac{1}{2}\int_{1/2}^{3/2} g(y)\,dy$$

$$= \frac{1}{4} + \frac{1}{2}\int_{1/2}^{1} dy = \frac{1}{2}.$$

b) Discuss the smoothness of u. Since f is not C^1, u also is not C^1. We now claim that even if g is not continuous, u is continuous. Indeed, since f is continuous $\frac{1}{2}[f(x+t)+f(x-t)]$ is continuous too. Given a sequence of points $(x_k, t_k) \longrightarrow (x,t)$ as $k \to +\infty$, then

$$\left| \frac{1}{2}\int_{x_k-t_k}^{x_k+t_k} g(y)dy - \frac{1}{2}\int_{x-t}^{x+t} g(y)\,dy \right| \le \frac{1}{2}\left|\int_{x_k-t_k}^{x-t} g(y)dy\right| +$$

$$+ \frac{1}{2}\left|\int_{x_k+t_k}^{x+t} g(y)\,dy\right| \longrightarrow 0 \quad \text{as } k \longrightarrow +\infty$$

From which we deduce that u is continuous ($u(x_k, t_k) \longrightarrow u(x,t)$ as $k \to +\infty$).

Still, u is not C^1. What can we say about the singularities of u? As noticed before, the singularities of the solution propagate along the characteristics.

How we find the singularities of u? We look at the point of singularity of the initial data f and g and we look at their evolution along the charac.

In our case, singularities are at points $\{-1, 0, 1\}$. This means that singularities can only live on these curves:

$$\{x+t=0\} \cup \{x-t=0\} \cup \{x+t=1\} \cup \{x-t=1\} \cup \{x+t=-1\} \cup \{x-t=-1\}.$$

$$\left(\text{i.e. } \{ x \pm t = 0, 1, -1 \} \right)$$

In our case we can at least say that u is a generalized solution.

Example 2.3. Consider the following Cauchy problem

$$\begin{cases} u_{tt} - 9u_{xx} = 0 \\ u(x,0) = f(x) = \begin{cases} 1 & \text{for } |x| \le 2 \\ 0 & \text{for } |x| > 2 \end{cases} \\ u_t(x,0) = g(x) = \begin{cases} 1 & \text{for } |x| \le 2 \\ 0 & \text{for } |x| > 2 \end{cases} \end{cases}$$

a) Find $u(0, \frac{1}{6})$. As before, we use the d'Alembert's formula

$$u(x,t) = \frac{f(x+3t) + f(x-3t)}{2} + \frac{1}{6} \int_{x-3t}^{x+3t} g(y)\, dy, \quad \text{at } x=0,\ t=\frac{1}{6}$$

$$u(0,\tfrac{1}{6}) = \frac{f(\frac{1}{2}) + f(-\frac{1}{2})}{2} + \frac{1}{6}\int_{-\frac{1}{2}}^{\frac{1}{2}} g(s)\,ds = \frac{7}{6}.$$ 6.10

b) Discuss the large time behaviour of the solution: for that, fix $\bar{x} \in \mathbb{R}$ and let t go to infinity:

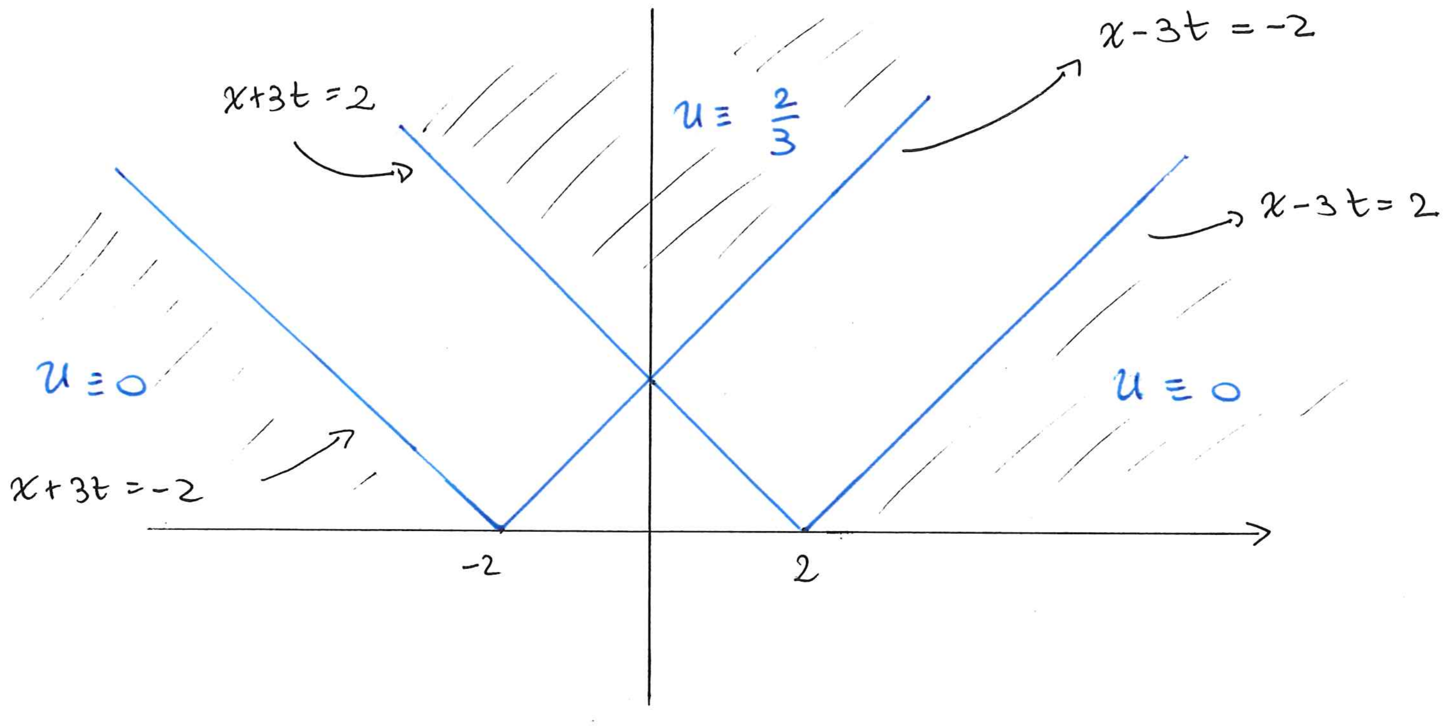
$$\lim_{t\to+\infty} u(\bar{x},t) = \lim_{t\to+\infty}\left[\frac{f(\bar{x}+3t)+f(\bar{x}-3t)}{2} + \frac{1}{6}\int_{\bar{x}-3t}^{\bar{x}+3t} g(y)\,dy\right]$$

For $\bar{x}$ fixed, if t is large then, $\bar{x}+3t > 2$ and $\bar{x}-3t < -2$, which means $f(\bar{x}+3t) = f(\bar{x}-3t) = 0$ and

$$\int_{\bar{x}-3t}^{\bar{x}+3t} g(y)\,dy = \int_{-2}^{2} dy = 4$$

Hence, for $x = \bar{x}$, if t is large $u(\bar{x},t) = \frac{1}{6}\cdot 4 = \frac{2}{3}$.

$$\lim_{t\to+\infty} u(x,t) = \frac{2}{3} \quad \forall x \in \mathbb{R}.$$



Long time behaviour of u