Recall of Lecture 6

We studied the 1D homogeneous wave equation and, using a suitable change of coordinates, we found the general solution of the 1D wave eq:

(1) $u_{tt} - c^2 u_{xx} = 0$ wave eq. 2nd order, linear, homogeneous

c = wave speed, $x \in \mathbb{R}$, $t \in \mathbb{R}^+$.

General solution:

(2) $u(x,t) = \underbrace{F(x+ct)}_{\text{Backward wave}} + \underbrace{G(x-ct)}_{\text{Forward wave}}$, with F, G smooth.

If $u(x,t)$ satisfies (2) with piecewise continuous functions F and G $\Rightarrow$ u is a generalized solution of the wave eq.

Then we studied the Cauchy problem for the homogeneous wave eq:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & (x,t) \in \mathbb{R} \times \mathbb{R}^+ \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

The solution is given by d'Alembert's formula:

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)\,dy$$

Remark: If u is smooth everywhere except at (x_0, t_0) then either F is not smooth at $x_0 + ct_0$ or G is not smooth at $x_0 - ct_0$ (or both). Thus, for any time $t_1 \neq t_0$, u is smooth except at $x^- - ct_1 = x_0 - ct_0$

$x^+ + ct_1 = x_0 + ct_0$.

The singularity of solutions of the wave equation are traveling only along characteristics.

Remark: For the wave equation in higher dimension

$$u_{tt} = c^2 \Delta u$$

there are similar formula to the d'Alembert's one but they are more complicated and they go beyond the scope of this class.

- Today we will study the non-homogeneous wave equation and we will see further properties of the wave equation.
- We will also see the method of separation of variables that is of great utility when solving second order linear equations

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Domain of dependence and region of influence (Section 4.4)

Let us consider the 1D homogeneous wave equation:

$$(1)\quad \begin{cases} u_{tt} - c^2 u_{xx} = 0 , & (x,t) \in \mathbb{R} \times (0,+\infty) \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

Q : How f and g influence the value of u at a given point (x_0, t_0)? How fast the information propagates?

The answer to the second question is suggested by the factorization of the solution in the sum of two traveling waves : $u(x,t) = F(x+ct) + G(x-ct)$, $c \in \mathbb{R}^+$.

The information propagates with finite speed c.

To answer to the first question we recall the d'Alembert's formula that gives us the value at (x_0, t_0):

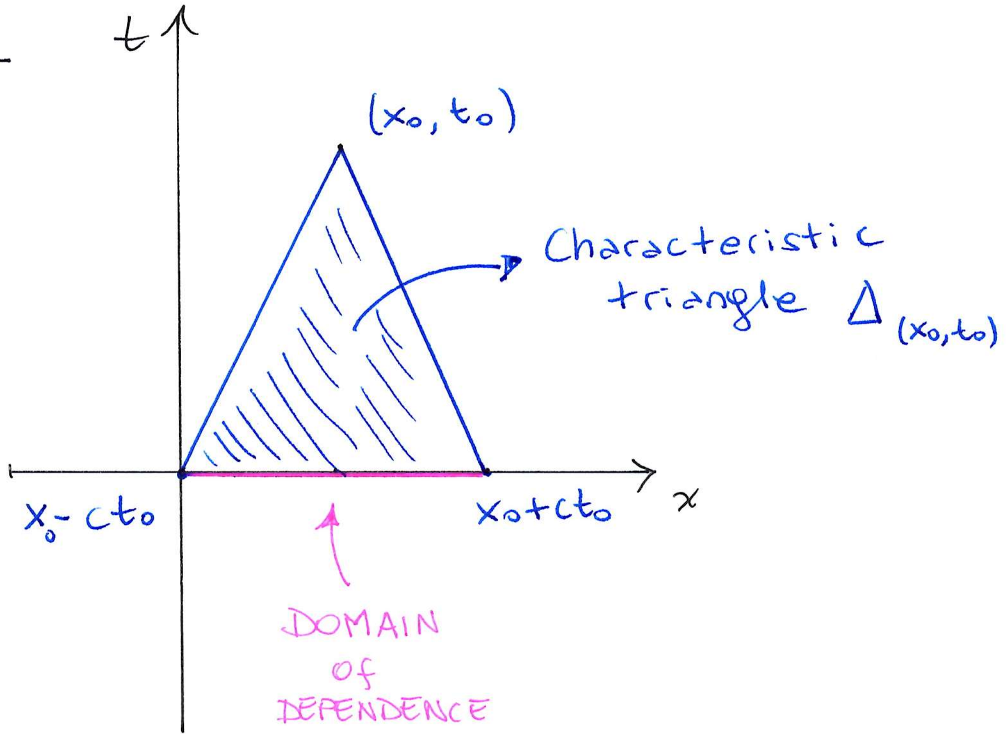
$$u(x_0, t_0) = \frac{f(x_0 + ct_0) + f(x_0 - ct_0)}{2} + \frac{1}{2c} \int_{x_0 - ct_0}^{x_0 + ct_0} g(s)\, ds$$

The value of u at the point (x_0, t_0) is determined by the values of f at the boundaries of the interval $[x_0 - ct_0, x_0 + ct_0]$ and by the value of g ALONG the interval. The interval $[x_0 - ct_0, x_0 + ct_0]$ is called domain of dependence of u at the point (x_0, t_0). If we change the initial data at points outside the interval, the value of the solution at point (x_0, t_0) will not change.

Also, fixing (x_0, t_0) consider the plane (x, t) and the characteristic lines (remember, information propagates along characteristics) passing through the point (x_0, t_0):

$$x - ct = x_0 - ct_0 \quad , \quad x + ct = x_0 + ct_0 .$$

These two lines intersect the x-axis at the points $(x_0 - ct_0, 0)$ and $(x_0 + ct_0, 0)$ respectively. The triangle formed by these lines and the interval $[x_0 - ct_0, x_0 + ct_0]$ is called characteristic triangle.



Remark: If the initial conditions are smooth on $[x_0 - ct_0, x_0 + ct_0]$ the solution itself will be smooth in the characteristic triangle $\Delta_{(x_0,t_0)}$.

Now we can ask ourself the dual question:

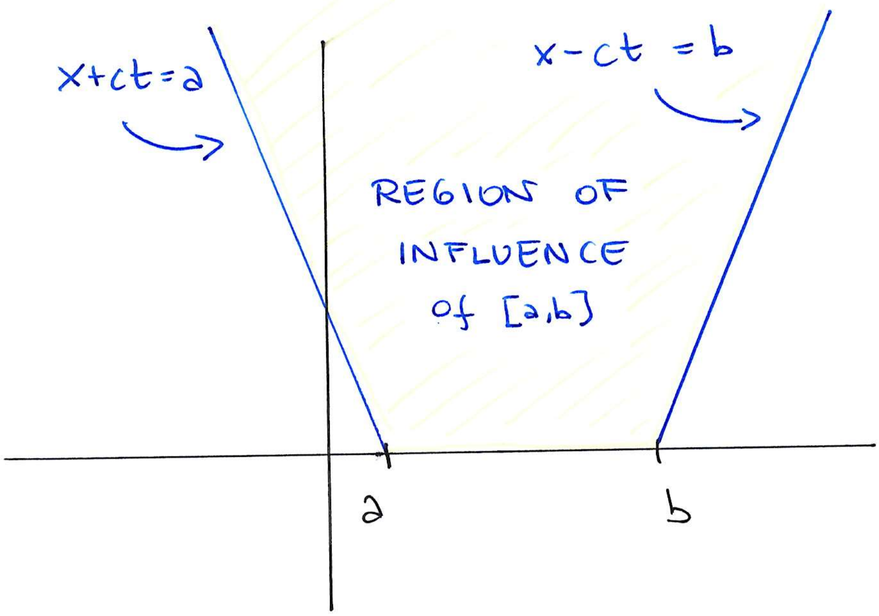
Q: which are the points in the half plane $t>0$ influenced by the initial data on a fixed interval $[a,b]$? The set of the points in the half plane that are affected by what happens to f and g in $[a,b]$ is called the region of influence of the interval $[a,b]$.

From d'Alembert's formula and the previous discussion we discover that the points in $[a,b]$ influence the value of u at a given point (x_0,t_0) if and only if

$$[x_0 - ct_0, \, x_0 + ct_0] \cap [a,b] \neq \emptyset .$$

Thus, the initial conditions along $[a,b]$ influence those points (x,t) which satisfy

$$x - ct \leq b \quad \text{and} \quad x + ct \geq a$$



Remark: If $g \equiv 0$ and $f \equiv 0$ outside $[a,b]$ then the solution u will be identically zero to the left of $x+ct=a$ and to the right of $x-ct=b$.

We are now ready for the non-homogeneous wave equation.

The Cauchy problem for the nonhomogeneous wave equation (Section 4.5)

The general nonhomogeneous 1D wave equation has the following form:

$$(2)\quad \begin{cases} u_{tt} - c^2 u_{xx} = F(x,t), & x \in \mathbb{R},\ t \in (0,+\infty) \\ u(x,0) = f(x), & x \in \mathbb{R} \\ u_t(x,0) = g(x), & x \in \mathbb{R} \end{cases}$$

This Cauchy problem models, for example, the vibration of an ideal string subject to an external force $F(x,t)$.
As in the homogeneous case, f and g are given functions that represent the shape and the vertical velocity of the string at time zero.

As for the homogeneous case, we wish to have an analogous of the d'Alembert's formula.

To do this, one integrates over the characteristic triangle T of a generic point (x_0,t_0):

$$\iint_{\Delta(x_0,t_0)} F(x,t)\,dx\,dt = \iint_{\Delta(x_0,t_0)} (u_{tt} - c^2 u_{xx})\,dx\,dt$$

Then after a series of computations (see Proposition 4.8, the proof contains all the computations, that are not examinable but you may find it interesting) one gets:

$$\boxed{u(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\xi,\tau)\,d\xi\,d\tau}$$

(Formula 4.17 in PR)

Theorem (Proposition 4.8, existence part)

The solution of the Cauchy problem (2) is given by:

$$u(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds + \frac{1}{2c}\int_0^t\int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\xi,\tau)\,d\xi\,d\tau .$$

That is the d'Alembert's formula for the nonhomogeneous wave equation.

Remark: Note that for $F=0$ the two d'Alembert's formula coincide. The value of u at (x_0,t_0) is given by the value of the initial data on the whole characteristic triangle.

Example 4.12 Consider the following problem:

$$\begin{cases} u_{tt} - 9u_{xx} = e^{x} - e^{-x}, & x\in\mathbb{R},\ t\in(0,+\infty) \\ u(x,0) = x, & x\in\mathbb{R} \\ u_t(x,0) = \sin x, & x\in\mathbb{R} \end{cases}$$

We now just apply the d'Alembert's formula with $c=3$.

$$u(x,t) = \frac{x+\cancel{3t}+x-\cancel{3t}}{2} + \frac{1}{6}\int_{x-3t}^{x+3t} \sin(s)\,ds + \frac{1}{6}\int_0^t\int_{x-3(t-\tau)}^{x+3(t-\tau)} (e^{\xi}-e^{-\xi})\,d\xi\,d\tau$$

$$u(x,t) = x + \frac{1}{6}\left[-\cos(s)\Big|_{s=x-3t}^{s=x+3t}\right] + \frac{2}{6}\int_0^t\int_{x-3(t-\tau)}^{x+3(t-\tau)} \sinh(\xi)\,d\tau$$

$$u(x,t) = x - \frac{1}{6}\left[-\cos(s)\Big|_{s=x-3t}^{s=x+3t}\right] + \frac{1}{3}\int_0^{\tau} \cosh(s)\Big|_{s=x-3(t-\tau)}^{s=x+3(t-\tau)} d\tau$$

$$\Rightarrow u(x,t) = x - \frac{1}{6}\left[\cos(x+3t) - \cos(x-3t)\right] +$$

$$+ \frac{1}{3}\int_0^{\tau}\left[\cosh(x+3t-3\tau) - \cosh(x-3t+3\tau)\right] d\tau$$

$$\left(\text{Recalling} \quad \cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right)$$

$$u(x,t) = x + \frac{1}{3}\sin(x)\sin(3t) + \frac{1}{3}\left[-\frac{1}{3}\sinh(x+3t-3\tau) - \frac{1}{3}\sinh(x-3t+3\tau)\right]$$

(evaluated at $\tau=0, \tau=t$)

$$\boxed{u(x,t) = x + \frac{1}{3}\sin(x)\sin(3t) + \frac{1}{9}\left[\sinh(x+3t) + \sinh(x-3t) - 2\sinh(x)\right]}$$

Remark : u is an odd function of x, is this a coincidence?

Example 4.14 (How to reduce a nonhomogeneous problem to an homogeneous one)

Consider the nonhomogeneous wave eq. given by:

$$\begin{cases} u_{tt} - u_{xx} = t^7 \\ u(x,0) = 2x + 2\sin x \\ u_t(x,0) = 0 \end{cases}$$

We could solve it using d'Alembert's formula (do it as an exercise!). However, in many cases it is possible to reduce a nonhomogeneous problem to an homogeneous one IF we can find a particular solution v of the given nonhomog. eq. This will eliminate the need to perform the double integral in d'Alembert's formula and this technique is very useful when F is very simple, such as $F = F(x)$ or $F = F(t)$.

Suppose we find a particular solution v, then we consider $w = u - v$. Since the wave eq. is LINEAR by SUPERPOSITION PRINCIPLE w will solve the following homogeneous pb:

$$\begin{cases} w_{tt} - w_{xx} = 0, & x \in \mathbb{R},\ t \in (0, +\infty) \\ w(x,0) = f(x) - v(x,0), & x \in \mathbb{R} \\ w_t(x,0) = g(x) - v_t(x,0), & x \in \mathbb{R} \end{cases}$$

Hence, w can be found using the d'Alembert's formula for the homogeneous problem.

The final solution will be given by : $\boxed{u = \underbrace{v}_{\text{particular solution}} + \underbrace{w}_{\text{solution of the associated homog. pb.}}}$

Since in our case $F = F(t) = t^7$, we look for a function v depending only on t: $v = v(t)$. Thus we need to solve

$$\boxed{v_{tt} = t^7}$$ (notice that $v_{xx} = 0$ because v doesn't depend on x)

Let us choose as a particular solution $v(t) = \frac{t^9}{72}$ (of course this solution is not unique because we did not impose any initial condition for v and v_t at time 0)

Now define $w(x,t) = u(x,t) - v(t)$, and write the PDE for w:

$$\begin{cases} w_{tt} - w_{xx} = u_{tt} - u_{xx} - v_{tt} = u_{tt} - u_{xx} - t^7 = 0 \\ w(x,0) = u(x,0) - v(0) = 2x + 2\sin x \\ w_t(x,0) = u_t(x,0) - v_t(0) = 0 \end{cases}$$

This is now an homogeneous wave eq. Let's apply d'Alembert formula

$$w(x,t) = \frac{2(x+t) + 2\sin(x+t) + 2(x-t) + 2\sin(x-t)}{2} =$$

$$= 2x + \sin(x+t) + \sin(x-t).$$

Recalling that $u(x,t) = w(x,t) + v(t)$, this gives:

$$\boxed{u(x,t) = 2x + \sin(x+t) + \sin(x-t) + \frac{t^9}{72}}$$

Theorem (Unique solution for the wave equation, Prop. 4.8 uniqueness part) The problem (2) has a unique solution.

Proof: the existence of a solution is given by the fact that the d'Alembert's formula gives us a solution (existence ✓)

For uniqueness. Suppose by contradiction that u_1, u_2 are solutions of (2). Then we define the difference $w = u_1 - u_2$.

This leads to the following equation for w:

$$\begin{cases} w_{tt} - c^2 w_{xx} = (u_1)_{tt} - c^2 (u_1)_{xx} - \left[(u_2)_{tt} - c^2 (u_2)_{tt}\right] = F(x,t) - F(x,t) = 0 \\ w(x,0) = u_1(x,0) - u_2(x,0) = f(x) - f(x) = 0 \\ w_t(x,0) = (u_1)_t(x,0) - (u_2)_t(x,0) = g(x) - g(x) = 0 \end{cases}$$

Therefore, the only solution to this equation is $w(x,t) = 0$
(In lecture 6 we showed that when the RHS is zero, the d'Alembert's formula gives us the unique solution to the problem.)

Therefore $u_1 = u_2$ as desired. $\square$

Let us now introduce another property of the wave equation, that will show that symmetry in Example 4.12 was not casual.

Theorem (symmetry of wave equations) Given a general non-homogeneous wave equation if the initial data f, g and the inhomogeneity F are EVEN (resp. ODD, PERIODIC) with respect to x, then the solution has the same symmetry.

7.9

<u>Proof</u> Let us consider the even case : $f(x) = f(-x)$, $g(x) = g(-x)$, $F(x,t) = F(-x,t)$.

We define $v(x,t) := u(-x,t)$, and we want to show that $v = u$ (i.e. the solution u is even w.r.t. x).

We note that :

·) $v_t(x,t) = u_t(-x,t)$, $v_{tt}(x,t) = u_{tt}(-x,t)$;

··) $v_x(x,t) = -u_x(-x,t)$, $v_{xx}(x,t) = u_{xx}(-x,t)$.

Therefore :

$$\begin{cases} v_{tt}(x,t) - c^2 v_{xx}(x,t) = u_{tt}(-x,t) - c^2 u_{xx}(-x,t) = \underbrace{F(-x,t) = F(x,t)}_{\text{by hypotheses on } F} \\ v(x,0) = u(-x,0) = f(-x) = f(x) \\ v_t(x,0) = u_t(-x,0) = g(-x) = g(x) \end{cases}$$

by hypotheses on f ang g

Thus v satisfies the same wave equation with the same boundary conditions as u, therefore by uniqueness $v = u$.

Let us now look at the odd case (this is a good EXERCISE)

By assumption :

1. $f(x) = -f(-x)$;
2. $g(x) = -g(-x)$;
3. $F(x,t) = -F(-x,t)$.

Define $v(x,t) = -u(-x,t)$, compute the derivatives in t and x :

$v_t(x,t) = -u_t(-x,t)$, $v_{tt}(x,t) = -u_{tt}(-x,t)$

$v_x(x,t) = u_x(-x,t)$, $v_{xx}(x,t) = -u_{xx}(-x,t)$. Thus

$$\begin{cases} v_{tt}(x,t) - c^2 v_{xx}(x,t) = -u_{tt}(-x,t) + c^2 u_{xx}(-x,t) = -F(-x,t) = F(x,t) \checkmark \\ v(x,0) = -u(-x,0) = -f(-x) = f(x) \checkmark \\ v_t(x,0) = -u_t(-x,0) = -g(-x) = g(x) \checkmark \end{cases} \Rightarrow v = u \text{ by uniqueness.}$$

Let now finish with the periodic case (EXERCISE)

By assumption there exists $L>0$ s.t :

1. $f(x) = f(x+L)$;
2. $g(x) = g(x+L)$;
3. $F(x,t) = F(x+L,t)$.

Define $v(x,t) = u(x+L,t)$ and, as before, plug $v(x,t)$ into the equation :

$$\begin{cases} v_{tt}(x,t) - c^2 u_{xx}(x,t) = u_{tt}(x+L,t) - c^2 u_{xx}(x+L,t) = F(x+L,t) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = F(x,t) \checkmark \\ v(x,0) = u(x+L,0) = f(x+L) = f(x) \checkmark \\ v_t(x,0) = u_t(x+L,0) = g(x+L) = g(x) \checkmark \end{cases}$$

Thus v fulfills the same equation as u and by uniqueness $v = u$. $\square$

An application: one-sided boundary conditions.

Consider the wave equation with an extra boundary condition on $x=0$:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & x>0, t>0 \\ u(x,0) = f(x), & x>0 \\ u_t(x,0) = g(x), & x>0 \\ u(0,t) = 0 & t\geq 0 \end{cases}$$

In order to fulfill the boundary condition $u(0,t)=0$ we extend f and g in an odd way :

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x\geq 0 \\ -f(-x) & \text{for } x<0 \end{cases}, \quad \tilde{g}(x) = \begin{cases} g(x) & \text{for } x\geq 0 \\ -g(-x) & \text{for } x<0 \end{cases}$$

Then one just solves the equation:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & , x \in \mathbb{R}^+, t>0 \\ u(x,0) = \tilde{f}(x) \\ u_t(x,0) = \tilde{g}(x) \end{cases}$$

The solution u will be odd in x because $\tilde{f}$ and $\tilde{g}$ are odd, therefore u will always satisfy $u(0,t) = 0$.
Indeed: $u(x,t) = -u(x,t) \Rightarrow u(0,t) = -u(0,t) \Rightarrow u(0,t) = 0$.

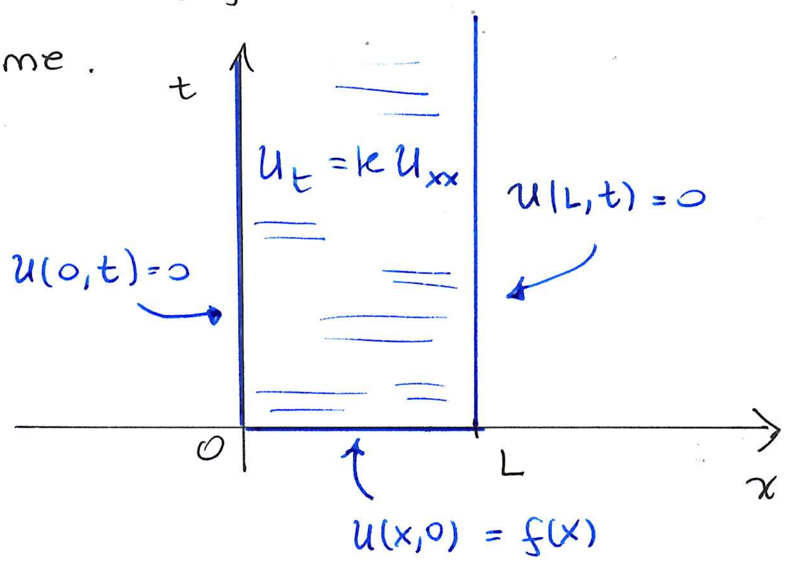
Separation of variables We now introduce the method of separation of variables to solve linear partial differential equations with boundary and/or initial conditions. Let's see this method directly in the case of the heat equation.

Consider the Cauchy problem:

$$\text{PDE} \rightsquigarrow \begin{cases} u_t - k u_{xx} = 0 & , x \in (0,L), t>0 \quad \text{(Second order, linear, homogeneous)} \\ u(0,t) = u(L,t) = 0 & , t>0 \rightsquigarrow \text{Boundary conditions} \\ u(x,0) = f(x), & x \in (0,L). \rightsquigarrow \text{initial condition: how the temperature is distributed at time zero.} \\ k \in \mathbb{R}^+ \text{ constant of diffusivity.} \end{cases}$$

The heat equation describes the diffusion of heat in a one-dimensional structure (for example a metal bar of length L) over time.

The boundary conditions are telling us that the boundary of the metal bar are kept at zero.



Let's now solve this problem using the method of separation of variables. The first step consists in seeking for a solution u that has the form of a product solution, or separate solution: 7.12

$$u(x,t) = X(x) \cdot T(t)$$

where $X : [0,L] \to \mathbb{R}$, $T : [0,+\infty) \to \mathbb{R}$.

Then we plug this in the heat equation:

$$T'(t)X(x) - k\,X''(x)T(t) = 0 \iff T'(t)X(x) = k\,X''(x)\,T(t)$$

$$\overset{(*)}{\iff} \quad \underbrace{\frac{T'(t)}{k\,T(t)}}_{\text{only dependent by } t} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{only dependent by } x} .$$

Therefore the only possibility for these two functions to be equal is to be constant. We call such constant $-\lambda$:

$$\frac{T'(t)}{k\,T(t)} = \frac{X''(x)}{X(x)} = -\lambda .$$

We are now left with two ODEs:

$$(3) \begin{cases} X''(x) = -\lambda X(x), & x \in (0,L) \\ T'(t) = -k\lambda T(t), & t > 0 \end{cases}$$

Remark Up to now we did not take into account the initial condition $u(x,0) = f(x)$. Obviously we are not interested in the identically zero solution $u(x,t) \equiv 0$, thus we look for functions X and T that do not vanish identically.

(*) This step is the actual SEPARATION of VARIABLES.

$$\text{ODEs}\begin{cases} X''(x) = -\lambda X(x), & x \in (0,L) \\ T'(t) = -k\lambda T(t), & t \in (0,+\infty) \end{cases}$$

These ODEs are only coupled by the separation constant $-\lambda$. Also, u satisfies the boundary conditions $u(0,t) = u(L,t) = 0$ if and only if :

$$u(0,t) = X(0)T(t) = 0 \quad \forall t > 0$$
$$u(L,t) = X(L)T(t) = 0 \quad \forall t > 0$$

The above conditions are fulfilled either if $T(t) \equiv 0$ (trivial solution) or if $X(0) = X(L) = 0$, that represents the interesting case.

Let now start from the ODE in X :

$$(4)\begin{cases} X''(x) = -\lambda X(x), & x \in (0,L) \\ X(0) = X(L) = 0 \end{cases}$$

Case 1 : $\lambda < 0$: The solution has the form :

$X(x) = \alpha \cosh(\sqrt{-\lambda}\, x) + \beta \sinh(\sqrt{-\lambda}\, x)$. From the boundary conditions we have : $X(0) = 0$, and since

$\sinh(0) = 0$, $\cosh(0) = 1$ we obtain $0 = X(0) = \alpha$.

Thus $X(x) = \beta \sinh(\sqrt{-\lambda}\, x)$.

From the condition $X(L) = 0$ we now get :

$$0 = X(L) = \beta \sinh(\sqrt{-\lambda}\, L)$$

and since $\sinh$ only vanishes at zero, $\sinh(\sqrt{-\lambda}\, L) \neq 0$ and then $\beta = 0$. Therefore the only solution compatible with the boundary conditions is the trivial one, and we discard this case.

Case 2 : $\lambda = 0$

In this case $X''(x) = 0$, that is $X(x) = \alpha + \beta x$.

Like before we use the boundary conditions to determine β and α:

$$X(0) = 0 \implies \alpha = 0$$
$$X(L) = 0 \implies \beta = 0.$$

Also here the only possible solution is the trivial one, and we discard this case.

Case 3 : $\lambda > 0$

Now the solution for X is:

$$X(x) = \alpha \cos(\sqrt{\lambda}\, x) + \beta \sin(\sqrt{\lambda}\, x).$$

We use the boundary conditions to find the values of α, β and λ:

$$\alpha = X(0) = 0 \implies \alpha = 0$$

Then $X(x) = \beta \sin(\sqrt{\lambda}\, x)$.

In order for X to satisfy $X(L) = 0$ we must have

$\sqrt{\lambda}\, L = n\pi$, $n \in \mathbb{N}$ (the sin vanishes only at multiples of π). From this we get

$$\lambda = \left(\frac{n\pi}{L}\right)^2.$$

Then $X(x) = \beta \sin(\sqrt{\lambda}\, x)$ is a solution compatible with the boundary conditions iff $\lambda = \left(\frac{n\pi}{L}\right)^2$, $n \in \mathbb{N}$.

Thus
$$X(x) = \beta \sin\left(\frac{n\pi}{L} x\right) \quad \forall\, n \in \mathbb{N}.$$

Note that the set of all solutions of (4) is an infinite sequence of functions

$$X_n(x) = \sin\left(\frac{n\pi}{L}x\right), \quad n \in \mathbb{N} \qquad \left(\lambda_n = \left(\frac{n\pi}{L}\right)^2, n \in \mathbb{N}\right)$$

Problem for T:

(5) $T'(t) = -k\lambda T(t)$, $t \in (0, +\infty)$

The general solution for (5) is $T(t) = Be^{-k\lambda t}$.

Since now we are conseder a sequence of λ_n we have:

$T_n(t) = B_n e^{-k\lambda_n t}$. Therefore the sequence of separated solutions is given by:

$$u_n(x,t) = X_n(x)\,T_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\lambda_n t}$$

Note that the heat equation is LINEAR, then by the SUPERPOSITION PRINCIPLE (see Lecture 1, 1.8 and 1.9)

[If $Lu = 0$ linear eq. $u_1, \dots, u_n$ solutions $\Rightarrow$ $\sum_{i=1}^{n} c_i u_i(x)$, $c_i \in \mathbb{R}$ is also a solution]

we have that any finite linear combination is still a solution to the heat equation:

$$\boxed{u(x,t) = \sum_{n=1}^{N} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\lambda_n t}}$$

$\hookrightarrow$ THIS SOLUTION ALSO SATISFIES THE BOUNDARY CONDITIONS

Now it is time to consider the initial condition.

If $f(x)$ admits the following Fourier expansion:

$$f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)$$

then a natural candidate for a solution is

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right) e^{-\lambda_n k t}$$

$$\left(\text{because} \quad u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) = f(x).\right)$$

We will see all these things more in detail next week.

Formally we solved
$$\begin{cases} u_t - k u_{xx} = 0 & x \in (0,L),\ t \in \mathbb{R}^+ \\ u(0,t) = u(L,t) = 0 & \\ u(x,0) = f(x) & x \in (0,L) \end{cases}$$

actually one should also verify that the final solution obtained is differentiable once with respect to t, twice w.r.t. x and we can differentiate the series term by term.