- In today's lecture we will study the method of separation of variables and we will apply it to the heat equation with Dirichlet boundary conditions, to the wave equation on an interval, and we will study different types of boundary conditions.

Heat equation: homogeneous boundary conditions

- Why heat equation? the heat equation is a linear second order PDE that describes how the distribution of heat evolves over time in a medium. Heat flows from places where it is higher towards places where it is lower(*). This equation was derived and solved by Joseph Fourier in 1822.

Heat equation in $\mathbb{R}^3$: $u = u(t,x,y,z)$

$$u_t = K(u_{xx} + u_{yy} + u_{zz}) = K\Delta u, \quad K \in \mathbb{R}$$

K is the diffusivity of the medium, the function u represents the temperature at (x,y,z) at time t.

The equation says that the rate u_t at which the material at a point (x,y,z) will heat up (or cool down) is proportional to how much hotter (or cooler) the surronding material is.

The heat equation arises in the modeling of a number of phenomena:
- in finantial mathematics, in the modeling of options;
- in probability theory it is connected with the study of Brownian motion
- in physics for modeling particle diffusion, etc.

(*) Second law of thermodynamics

(Section 5.2 and 5.3)

- Heat equation with homogeneous boundary conditions.

Consider the following Cauchy problem associated to the 1D heat equation:

$$\begin{cases} u_t - K u_{xx} = 0 & (x,t) \in [0,L] \times (0,+\infty) \\ u(0,t) = u(L,t) = 0 \, , & t \in [0,+\infty) \xrightarrow{} \text{Dirichlet boundary conditions} \\ \underbrace{u(x,0) = f(x)}_{\text{Initial condition}} & x \in [0,L] \end{cases}$$

$K > 0$, f initial condition.

Remark: in order to have compatibility between boundary and initial conditions we assume the following compatibility conditions: $f(0) = f(L) = 0$.

- The Cauchy problem above corresponds to the evolution in time of the temperature in a 1D metal bar of length L, whose initial temperature is known to be equal to f.
- The Dirichlet boundary conditions tell us that the two ends of the bar are at temperature zero for all times.
- No internal source of heat is considered.
- The Cauchy problem above is also called an initial boundary problem (and it is homogeneous).

Let's consider solutions of the heat equation that satisfy the boundary conditions and that have the form:

$$u(x,t) = X(x)T(t).$$

Note that at this step we are not asking that u satisfies the initial condition.

As last week, by plugging the solution $u = X(x)T(t)$ in the PDE we obtain:

$$X T_t = k X_{xx} T$$

We now perform the separation of variables:

$$\underbrace{\frac{T_t}{kT}}_{\text{depends only on } t} = \underbrace{\frac{X_{xx}}{X}}_{\text{depends only on } x} = -\lambda$$

(λ is called separation constant)

Therefore we have the following ODEs:

$$\begin{cases} X_{xx} = -\lambda X & x \in (0,L) \\ T_t = -\lambda k T & t > 0 \end{cases}$$

$\rightsquigarrow$ those ODEs are coupled by the constant λ

Recall the boundary conditions: $u(0,t) = u(L,t) = 0$, this implies:

$$u(0,t) = X(0)T(t) = 0 \quad \forall t > 0$$

$$u(L,t) = X(L)T(t) = 0 \quad \forall t > 0$$

When are these conditions satisfied? Either $T(t) \equiv 0$ (trivial solution), or $X(0) = X(L) = 0$ (interesting case).

Problem for X:

$$\begin{cases} X_{xx} = -\lambda X & x \in (0,L) \\ X(0) = X(L) = 0 \end{cases}$$

$\leftrightsquigarrow$ This is also called EIGENVALUE PROBLEM

It is known that the general solution of the second order linear ODE $X_{xx} = -\lambda X$ is of the form:

1. $X(x) = \alpha \cosh(\sqrt{-\lambda}\, x) + \beta \sinh(\sqrt{-\lambda}\, x)$ with $\alpha, \beta \in \mathbb{R}$, $\lambda < 0$
2. $X(x) = \alpha + \beta x$ with $\alpha, \beta \in \mathbb{R}$, $\lambda = 0$
3. $X(x) = \alpha \cos(\sqrt{\lambda}\, x) + \beta \sin(\sqrt{\lambda}\, x)$, $\alpha, \beta \in \mathbb{R}$, $\lambda > 0$

Let's study one by one these cases:

Negative eigenvalue $\lambda < 0$. The solution is $X(x) = \alpha \cosh(\sqrt{-\lambda}\, x) + \beta \sinh(\sqrt{-\lambda}\, x)$ By imposing $X(0) = X(L) = 0$ we get $\alpha = \beta = 0$.

Zero eigenvalue $\lambda = 0$: By imposing the boundary conditions to $X(x) = \alpha + \beta x$ we get $\alpha = \beta = 0$.

Positive eigenvalue $\lambda > 0$: The condition $X(0)$ implies $\alpha = 0$. The condition $X(L) = 0$ implies that $\beta \sin(\sqrt{\lambda} L)$ is zero $\Rightarrow$ either $\beta = 0$ or $\sin(\sqrt{\lambda} L) = 0$. The latter case is the only non trivial one and leads to the condition $\sqrt{\lambda} L = n\pi$, $n \in \mathbb{N}$. Hence λ is an eigenvalue if and only if:

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \quad n \in \mathbb{N}.$$

The corresponding eigenfunctions are:

$$X_n(x) = \sin\left(\frac{n\pi}{L} x\right)$$ and they are defined up to a multiplicative constant.

Thus the set of all solutions of :

$$\begin{cases} X_{xx} = -\lambda X, & x \in (0,L) \\ X(0) = X(L) = 0 \end{cases}$$

is an infinite sequence of eigenfunctions X_n, each associated with a positive eigenvalue λ_n :

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \ n \in \mathbb{N}, \quad X_n = \sin(\sqrt{\lambda_n}\, x).$$

Problem for T : The general solution of $T_t = -k\lambda T$ is $T(t) = B\, e^{-k\lambda T}$. Substituting λ_n we obtain the sequence of solutions : $T_n(t) = B_n e^{-k\lambda_n t}$.

Therefore the sequence of separated solutions is :

$$u_n(x,t) = X_n(x)\, T_n(t) = B_n \sin(\sqrt{\lambda_n}\, x)\, e^{-k\lambda_n t}.$$

Since the heat equation is linear we can apply the superposition principle, therefore any linear combination

$$u(x,t) = \sum_{n=1}^{+\infty} B_n \sin(\sqrt{\lambda_n}\, x)\, e^{-k\lambda_n t}$$

is still a solution of the heat equation with Dirichlet boundary conditions. Now we need to consider the initial condition $u(x,0) = f(x)$.

By Fourier theory (and some considerations in the EXTRA), f is a linear combination of eigenfunctions :

$$f(x) = \sum_{n=1}^{+\infty} C_n \sin(\sqrt{\lambda_n}\, x)$$

Now we have the general solution:

$$u(x,t) = \sum_{n=1}^{+\infty} B_n \sin(\sqrt{\lambda_n}\, x)\, e^{-k\lambda_n t}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

with the initial condition:

$$f(x) = \sum_{n=1}^{+\infty} C_n \sin(\sqrt{\lambda_n}\, x).$$

The coefficients $B_n \in \mathbb{R}$ have to be recovered from the initial condition.

Indeed, recalling that $u(x,0) = f(x)$ we discover that

$$u(x,0) = \sum_{n=1}^{+\infty} B_n \sin(\sqrt{\lambda_n}\, x) = f(x) \implies B_n = C_n \quad \forall n \in \mathbb{N}.$$

We now need to determine the coefficients B_n: fix $m \in \mathbb{N}$ and multiply the expansion for $f(x)$ by $\sin\left(\frac{m\pi}{L} x\right)$

$$f(x) \sin\left(\frac{m\pi}{L} x\right) = \sum_{n=1}^{+\infty} B_n \sin\left(\frac{n\pi}{L} x\right) \sin\left(\frac{m\pi}{L} x\right)$$

Integrate over $[0,L]$:

$$\int_0^L f(x) \sin\left(\frac{m\pi}{L} x\right) dx = \sum_{n=1}^{+\infty} B_n \int_0^L \sin\left(\frac{n\pi}{L} x\right) \sin\left(\frac{m\pi}{L} x\right) dx$$

Since

$$\int_0^L \sin\left(\frac{n\pi}{L} x\right) \sin\left(\frac{m\pi}{L} x\right) dx = \begin{cases} L/2 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

we have that

$$\int_0^L f(x) \sin\left(\frac{m\pi}{L} x\right) dx = B_m \frac{L}{2}$$

and this gives:

$$\boxed{B_m = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{m\pi}{L} x\right) dx, \quad \forall m \in \mathbb{N}}.$$

Therefore, the coefficients B_m are uniquely determined from the initial condition.

Example 5.1 Consider the Cauchy problem:

$$\begin{cases} u_t - u_{xx} = 0 & (x,t) \in [0,\pi] \times [0,+\infty) \\ u(0,t) = u(\pi,t) = 0 & \forall\, t \geq 0 \\ u(x,0) = f(x) = \begin{cases} x & \text{for } x \in [0,\frac{\pi}{2}] \\ \pi - x & \text{for } x \in [\frac{\pi}{2},\pi] \end{cases} \end{cases}$$

This is again a 1D heat equation with Dirichlet boundary conditions in $[0\ \pi]$. Therefore we know that the general solution is:

$$u(x,t) = \sum_{n=1}^{+\infty} B_n \sin(nx)\, e^{-n^2 t} \qquad (\text{now } L = \pi)$$

Imposing $u(x,0) = f(x)$ we notice, as before, that the coeffients B_n are the Fourier coefficients of $f(x)$ that can be computed as follows:

$$B_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x\sin(nx)\,dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x)\sin(nx)\,dx\right]$$

$$= \frac{2}{\pi}\left[-\frac{x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\Bigg|_0^{\pi/2}\right] + \frac{2}{\pi}\left[-\frac{(\pi-x)\cos(nx)}{n} - \frac{\sin nx}{n^2}\Bigg|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{4}{\pi n^2}\sin\left(\frac{n\pi}{2}\right),$$

but

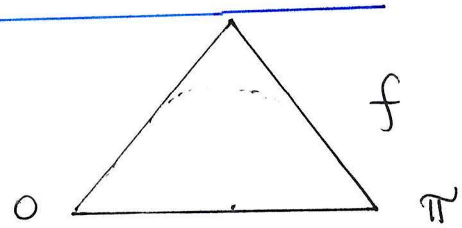
$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & \text{if } n = 2j \\ (-1)^{j+1} & \text{if } n = 2j-1 \end{cases}, \quad n \in \mathbb{N},\ j \in \mathbb{N}$$

The solution is given by:

$$u(x,t) = \sum_{j=1}^{+\infty} \frac{4}{\pi(2j-1)^2} (-1)^{j+1} \sin((2j-1)x)\, e^{-(2j-1)^2 t}$$

Figure :

Initial condition for the problem:



Boundary conditions: In the following we will encounter three types of boundary conditions:

- Dirichlet : $u(0,t) = u(L,t) = 0$
- Neumann : $u_x(0,t) = u_x(L,t) = 0$
- Mixed type / Robin : $\alpha u(0,t) + \beta u_x(0,t) = \gamma$ and $\delta u(L,t) + \varepsilon u_x(L,t) = \xi$

Separation of variables for the wave equation:

Let's now apply the method of separation of variables to solve the 1D wave equation with Neumann boundary conditions.

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & (x,t) \in [0,L] \times [0,+\infty) \rightsquigarrow \text{wave eq.} \\ u_x(0,t) = u_x(L,t) = 0 \ , \ t > 0 & \rightsquigarrow \text{Neumann boundary conditions} \\ u(x,0) = f(x),\ x \in \mathbb{R} & \\ u_t(x,0) = g(x),\ x \in \mathbb{R} & \end{cases} \Big\} \rightsquigarrow \text{Initial conditions}$$

As before we look for a solution of the form:

$u(x,t) = X(x)T(t)$, $(u \neq 0)$. At this stage we don't take into account the initial conditions. Differentiating in x and t:

$u_{tt} = X(x)T''(t)$, $u_{xx} = X''(x)T(t)$. Plugging into the eq:

we obtain :

$$X(x)\,T''(t) = c^2 X''(x)\,T(t)$$

(equivalent to : $X(x)\,T_{tt}(t) = c^2 X_{xx}(x)\,T(t)$)

By separation of variables we obtain:

$$\frac{T_{tt}(t)}{c^2 T(t)} = \frac{X_{xx}(x)}{X(x)} = -\lambda$$

Therefore we have the following ODEs :

$$\begin{cases} X_{xx}(x) = -\lambda X(x), & X_x(0) = X_x(L) = 0 \\ T''(t) = -c^2 \lambda T(t) \end{cases}$$

The general solution for the ODE in X is the following :

$$X(x) = \begin{cases} \alpha \cosh(\sqrt{-\lambda}\, x) + \beta \sinh(\sqrt{-\lambda}\, x), & \lambda < 0 \\ \alpha + \beta x & \lambda = 0 \\ \alpha \cos(\sqrt{\lambda}\, x) + \beta \sin(\sqrt{\lambda}\, x) & \lambda > 0 \end{cases}$$

In this case we need to impose the Neumann boundary conditions $X_x(0) = X_x(L) = 0$. We have to study the latter cases one by one (Exercise)

$$X_x(x) = \begin{cases} \sqrt{-\lambda}\left[\alpha \sinh(\sqrt{-\lambda}\, x) + \beta \cosh(\sqrt{-\lambda}\, x)\right], & \lambda < 0 \\ \beta & \lambda = 0 \\ \sqrt{\lambda}\left[-\alpha \sin(\sqrt{\lambda}\, x) + \beta \cos(\sqrt{\lambda} x)\right], & \lambda > 0 \end{cases}$$

Negative eigenvalue ($\lambda < 0$): $X_x(0) = 0 \implies \beta = 0$. Then $X_x(L) = 0 \implies \alpha \sinh(\sqrt{-\lambda}\, L) = 0^{(*)}$. Therefore $X(x) \equiv 0$ and for $\lambda < 0$ we didn't find non-trivial solutions.

($(*) \implies \alpha = 0$)

Zero eigenvalue ($\lambda = 0$). The only nontrivial solution is given by $X_0(x) = \alpha$.

Positive eigenvalue ($\lambda > 0$). Substituting the boundary conditions we find $\beta = 0$ and $\sin(\sqrt{\lambda} L) = 0$. Thus $\sqrt{\lambda} L = n\pi$ where $n \in \mathbb{N}$. Thus, $\lambda_n = \left(\frac{\pi n}{L}\right)^2$, $n \in \mathbb{N}$ are eigenvalues and the corresponding eigenfunctions are

$$X_n(x) = \alpha_n \cdot \cos\left(\frac{n\pi}{L} x\right).$$

Considering now the ODE for T : $T''(t) = -c^2 \lambda T(t)$.
If $\lambda = 0$, $T''_0(t) = 0$. Thus $T_0(t) = \gamma_0 + \delta_0 t$ and

$T''_n = -c^2 \left(\frac{n\pi}{L}\right)^2 T_n$. Therefore:

$$T_n(t) = \gamma_n \cos\left(\frac{cn\pi}{L} t\right) + \delta_n \sin\left(\frac{cn\pi}{L} t\right).$$

In conclusion, the general solution for the 1D wave equation with Neumann boundary conditions can be written as:

$$u(x,t) = \sum_{n=0}^{+\infty} X_n(x) T_n(t) = \frac{A_0 + B_0 t}{2} + \qquad (*)$$

$$+ \sum_{n=1}^{+\infty} \cos\left(\frac{n\pi}{L} x\right)\left(A_n \cos\left(\frac{n\pi c}{L} t\right) + B_n \sin\left(\frac{cn\pi}{L} t\right)\right)$$

(The factor ½ in the first term is just for convenience)

To find A_n we evaluate the function at the initial condition:

$$f(x) = u(x,0) = \frac{A_0}{2} + \sum_{n=1}^{+\infty} A_n \cos\left(\frac{n\pi}{L} x\right)$$

Fix $m \in \mathbb{N}$, multiply both sides above by $\cos\left(\frac{m\pi}{L}x\right)$ and integrate over $[0,L]$:

$$\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)dx = \int_0^L \frac{A_0}{2}\cos\left(\frac{m\pi}{L}x\right)dx \;+$$

$$+\sum_{n=1}^{+\infty} A_n \int_0^L \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx$$

Since $\int_0^L \cos\left(\frac{m\pi}{L}x\right)dx = \begin{cases} L & \text{if } m=0 \\ 0 & \text{if } m \geq 1 \end{cases}$ and

$$\int_0^L \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx = \begin{cases} L/2 & \text{if } n=m\neq 0 \\ 0 & \text{if } n\neq m \end{cases}$$

we obtain:

$$\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)dx \overset{(m=0)}{=} \int_0^L f(x)\,dx = \frac{A_0 L}{2}$$

$$\Rightarrow A_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$A_m = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)dx \qquad \forall m \geq 1.$$

$$\left(A_m = \frac{\int_0^L \cos\left(\frac{m\pi x}{L}\right) f(x)\,dx}{\int_0^L \cos^2\left(\frac{m\pi x}{L}\right)dx} \right)$$

The same procedure can be implemented to find the coefficients B_m:

$$g(x) = u_t(x,0) = \frac{B_0}{2} + \sum_{n=1}^{+\infty} \cos\left(\frac{n\pi}{L}x\right) B_n \left(\frac{n\pi c}{L}\right)$$

$$\Rightarrow \quad B_0 = \frac{2}{L}\int_0^L g(x)\,dx \quad , \quad B_m = \frac{2}{cm\pi}\int_0^L g(x)\cos\left(\frac{m\pi}{L}x\right)dx$$

for $m \geq 1$. Thus the problem is formally solved.

Example Consider the wave equation:

$$\begin{cases} u_{tt} - 4u_{xx} = 0 & (x,t) \in [0,1] \times [0,+\infty) \\ u_x(0,t) = u_x(1,t) = 0 & t \geq 0 \\ u(x,0) = f(x) = \frac{1}{2} + \frac{1}{2}\cos(2\pi x) , & x \in \mathbb{R} \\ u_t(x,0) = g(x) = 0 & x \in \mathbb{R} \end{cases}$$

Due to its form we can apply (*) and obtain:

$$u(x,t) = \frac{A_0 + B_0 t}{2} + \sum_{n=1}^{+\infty} \cos(n\pi x)\left[A_n \cos(2n\pi t) + B_n \sin(2n\pi t)\right]$$

To find the Fourier coefficients we impose the initial conditions:

$$\frac{1}{2} + \frac{1}{2}\cos(2\pi x) = u(x,0) = \frac{1}{2}A_0 + \sum_{n=1}^{+\infty} A_n \cos(n\pi x) .$$

It follows immediately that $A_0 = 1$, $A_2 = \frac{1}{2}$ and $A_n = 0$ otherwise. Using that $g(x) = 0$ we obtain $B_n \equiv 0$ $\forall n$. The solution to the Cauchy problem is therefore:

$$u(x,t) = \frac{1}{2} + \frac{1}{2}\cos(2\pi x)\cos(4\pi t)$$

0.1 Extra - Fourier Series

Given a function $g : [-L, L] \to \mathbb{R}$ (or equivalently $g : [0, 2L] \to \mathbb{R}$), the classical Fourier theory tells us that it can be written as

$$g(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right),$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} g(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} g(x)\,\cos\left(\frac{n\pi}{L}x\right)\,dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} g(x)\,\sin\left(\frac{n\pi}{L}x\right)\,dx.$$

So, why can we use only sin or only cos in our case?

Note that in our case our functions are defined on $[0, L]$ and we are using $\cos\left(\frac{n\pi}{L}x\right)$ and $\sin\left(\frac{n\pi}{L}x\right)$ to expand our functions, while the classical theory gives the expansion above for functions on $[-L, L]$. So, we can use the following trick: given a function $f : [0, L] \to \mathbb{R}$, let f^{odd} denotes the odd extension of f on $[-L, L]$, and f^{even} it even extension:

$$f^{odd}(x) := \begin{cases} f(x) & x \in (0, L] \\ f(0) = 0 & \\ -f(-x) & x \in [-L, 0) \end{cases}, \qquad f^{even}(x) := \begin{cases} f(x) & x \in [0, L] \\ f(-x) & x \in [-L, 0] \end{cases}$$

Now, writing f^{odd} using Fourier series, we have

$$f^{odd}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right),$$

where

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f^{odd}(x)\,dx, \\ a_n &= \frac{1}{L}\int_{-L}^{L} f^{odd}(x)\,\cos\left(\frac{n\pi}{L}x\right)\,dx, \\ b_n &= \frac{1}{L}\int_{-L}^{L} f^{odd}(x)\,\sin\left(\frac{n\pi}{L}x\right)\,dx. \end{aligned}$$

By the definition of f^{odd} we have

$$\begin{aligned} \int_{-L}^{L} f^{odd}(x)\,dx &= \int_{-L}^{0} f^{odd}(x)\,dx + \int_{0}^{L} f^{odd}(x)\,dx \\ &= -\int_{-L}^{0} f(-x)\,dx + \int_{0}^{L} f(x)\,dx \\ &= -\int_{0}^{L} f(y)\,dy + \int_{0}^{L} f(x)\,dx = 0, \end{aligned}$$

where in the third equality we used the change of variable $y = -x$. Hence $a_0 = 0$.

Analogously,

$$\int_{-L}^{L} f^{odd}(x) \cos\left(\frac{n\pi}{L}x\right) dx = -\int_{-L}^{0} f(-x) \cos\left(\frac{n\pi}{L}x\right) dx + \int_{0}^{L} f(x) \cos\left(\frac{n\pi}{L}x\right) dx$$
$$= -\int_{0}^{L} f(y) \cos\left(\frac{n\pi}{L}y\right) dy + \int_{0}^{L} f(x) \cos\left(\frac{n\pi}{L}x\right) dx = 0,$$

where we used that $\cos\left(\frac{n\pi}{L}y\right) = \cos\left(-\frac{n\pi}{L}y\right)$. Hence $a_n = 0$ for every $n \geq 1$

In the same way, since $\sin\left(-\frac{n\pi}{L}x\right) = -\sin\left(\frac{n\pi}{L}x\right)$

$$\int_{-L}^{0} f^{odd}(x) \sin\left(\frac{n\pi}{L}x\right) dx = \int_{0}^{L} f^{odd}(x) \sin\left(\frac{n\pi}{L}x\right) dx,$$

therefore

$$b_n = \frac{2}{L}\int_{0}^{L} f^{odd}(x) \sin\left(\frac{n\pi}{L}x\right) dx = \frac{2}{L}\int_{0}^{L} f(x) \sin\left(\frac{n\pi}{L}x\right) dx.$$

Thus, for $x \in [0, L]$ we have

$$f(x) = f^{odd}(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right), \qquad b_n = \frac{2}{L}\int_{0}^{L} f(x) \sin\left(\frac{n\pi}{L}x\right) dx,$$

which shows that f can be written only using $\sin\left(\frac{n\pi}{L}x\right)$.

Viceversa, if we consider f^{even} we see that

$$f^{even}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right),$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f^{even}(x)\, dx,$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f^{even}(x) \cos\left(\frac{n\pi}{L}x\right) dx,$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f^{even}(x) \sin\left(\frac{n\pi}{L}x\right) dx.$$

But since f^{even} is even and $f^{even} = f$ for $x > 0$, arguing as before one deduce that $b_n = 0$ for every n. In addition

$$\int_{-L}^{0} f^{even}(x)\, dx = \int_{0}^{L} f^{even}(x)\, dx,$$
$$\int_{-L}^{0} f^{even}(x) \cos\left(\frac{n\pi}{L}x\right) dx = \int_{0}^{L} f^{even}(x) \cos\left(\frac{n\pi}{L}x\right) dx,$$

therefore

$$a_0 = \frac{1}{L}\int_{0}^{L} f(x)\, dx, \qquad a_n = \frac{2}{L}\int_{0}^{L} f(x) \cos\left(\frac{n\pi}{L}x\right) dx.$$

Thus, for $x \in [0, L]$ we have

$$f(x) = f^{even}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} x\right)$$

with

$$a_0 = \frac{1}{L} \int_0^L f(x)\, dx, \qquad a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi}{L} x\right) dx,$$

which shows that f can be written only using constants and $\cos\left(\frac{n\pi}{L} x\right)$.

This proves that, given a function $f : [0, L] \to \mathbb{R}$, we can expand it using:
- either only $\sin\left(\frac{n\pi}{L} x\right)$;
- or only constants and $\cos\left(\frac{n\pi}{L} x\right)$.