- Today we will continue with the study of the method of separation of variables for homogeneous and inhomogeneous PDEs.
- We will also study the energy method to prove uniqueness of solutions.

Up to now we studied the method of separation of variables in the case of homogeneous heat and wave equations. Let's summarize it again before studying the non homogeneous case.

Recall of separation of variables for homogeneous PDEs:

1. Heat equation:

$$\begin{cases} u_t = K u_{xx} & (x,t) \in [0,L] \times [0,+\infty) \rightsquigarrow \text{Heat equation: 2nd order, linear, homogeneous.} \\ u(0,t) = u(L,t) = 0 & , t \in \mathbb{R}^+ \rightsquigarrow \text{Dirichlet boundary conditions} \\ u(x,0) = f(x) & , x \in [0,L] \rightsquigarrow \text{Initial condition} \end{cases}$$

Step 1: We look for solutions of the form: $u(x,t) = X(x)T(t)$. Differentiate and plug in the heat equation: $T_t(t)X(x) = K\,T(t)\,X_{xx}(x)$

"Separate" the equation: $$\underbrace{\frac{T_t(t)}{K\,T(t)}}_{\text{depends only on } t} = \underbrace{\frac{X_{xx}(x)}{X(x)}}_{\text{depends only on } x} = -\lambda \in \mathbb{R}$$

and couple with the boundary conditions: $X(0) = X(L) = 0$.
Ignore for now the initial condition and solve the ODE system:

$$(1)\ \begin{cases} X_{xx}(x) = -\lambda X(x) \\ X(0) = X(L) = 0 \end{cases}$$

and

$$(2)\ \{ T_t(t) = -k\lambda T(t)$$

The equation $X_{xx}(x) = -\lambda X(x)$ has the following solutions:

(i) $\lambda < 0$: $X(x) = \alpha \cosh(\sqrt{-\lambda}\, x) + \beta \sinh(\sqrt{-\lambda}\, x)$;

(ii) $\lambda = 0$: $X(x) = \alpha + \beta x$;

(iii) $\lambda > 0$: $X(x) = \alpha \cos(\sqrt{\lambda}\, x) + \beta \sin(\sqrt{\lambda}\, x)$.

Plugging in (i), (ii), (iii) the boundary conditions we obtain the following nontrivial solutions:

$$X_n(x) = \alpha_n \sin\left(\frac{n\pi}{L} x\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n \in \mathbb{N}$$

(the fact that we have a superposition of sine functions depends by the boundary conditions. As an exercise try to solve the heat equation with Neumann b.c.)

From (2) we have : $T_n(t) = e^{-\lambda_n k t}\, T_n(0)$.

Therefore, by the superposition principle we obtain the general solution of the heat equation with Dirichlet b.c. :

$$u(x,t) = \sum_{n=1}^{+\infty} B_n\, e^{-k\lambda_n t} \sin\left(\frac{n\pi}{L} x\right).$$

Then, by the initial condition we have:

$$u(x,0) = \sum_{n=1}^{+\infty} B_n \sin\left(\frac{n\pi}{L} x\right) = f(x)$$

and from this we deduce that the coefficients B_n are precisely the Fourier coefficients of f. This gives us a formal solution to our boundary/initial Cauchy problem.

The wave equation with Neumann's boundary conditions:

$$\begin{cases} u_{tt} = c^2 u_{xx} & , x \in [0,L], t \in \mathbb{R}^+ \\ u_x(0,t) = u_x(L,t) = 0 & , t \in \mathbb{R}^+ \\ u(x,0) = f(x) , & x \in [0,L] \\ u_t(x,0) = g(x) , & x \in [0,L] \end{cases}$$

$\rightsquigarrow$ homogeneous wave eq. 2nd order, linear.

$\rightsquigarrow$ Neumann's boundary conditions

Initial conditions

As before, we look for solutions $u(x,t) = X(x)T(t)$. We differentiate and plug into the wave equation:

$T_{tt}(t) X(x) = c^2 T(t) X_{xx}(x)$. We separate the equation in order to have a term that only depends on t and a term that only depends on x:

$$\frac{T_{tt}(t)}{c^2 T(t)} = \frac{X_{xx}(x)}{X(x)} = -\lambda \in \mathbb{R}$$

with Neumann boundary conditions

$$X_x(0) = X_x(L) = 0 .$$

[Rmk: $X_x(x) = X'(x)$ and $T_t(x) = T'(t)$]

We now have the ODEs:

$$\begin{cases} X_{xx}(x) = -\lambda X(x) , & X_x(0) = X_x(L) = 0 \quad \text{and} \\ T_{tt}(t) = -c^2 \lambda T(t) . \end{cases}$$

Now, plugging the Neumann b.c. we obtain that $X_n(x)$ will be of the form:

$$X_n(x) = \alpha_n \cos\left(\frac{n\pi}{L} x\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n \in \mathbb{N} .$$

Remark: we now have a cosines expansion instead of a sines one, because of the Neumann boundary conditions. This would be the case also for the heat equation with Neumann b.c.

Note also that for $\lambda_0 = 0$, $X_0(x) = \alpha_0 \in \mathbb{R}$.

From the equation $T_{tt}(t) = -\lambda c^2 T(t)$ we have:

$$T_n(t) = a_n \sin\left(\frac{n\pi c}{L} t\right) + b_n \cos\left(\frac{n\pi c}{L} t\right) \quad \text{for } n > 0$$

and $T_0(t) = a_0 + b_0 t$ for $n = 0$.

The general solution is:

$$u(x,t) = \frac{A_0 + B_0 t}{2} + \sum_{n=1}^{+\infty} \left[A_n \cos\left(\frac{n\pi c}{L} t\right) + B_n \sin\left(\frac{n\pi c}{L} t\right)\right] \cos\left(\frac{n\pi x}{L}\right)$$

Then, thanks to the initial conditions we have:

$$u(x,0) = \frac{A_0}{2} + \sum_{n=1}^{+\infty} A_n \cos\left(\frac{n\pi}{L} x\right) = f(x);$$

$$u_t(x,0) = \frac{B_0}{2} + \sum_{n=1}^{+\infty} \left(\frac{n\pi c}{L}\right) B_n \cos\left(\frac{n\pi x}{L}\right) = g(x).$$

From this we deduce that:

$$A_m = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{m\pi x}{L}\right) dx, \quad m \geq 0;$$

$$B_m = \frac{2}{c m\pi} \int_0^L g(x) \cos\left(\frac{m\pi}{L} x\right) dx, \quad m > 0;$$

$$B_0 = \frac{2}{L} \int_0^L g(x)\, dx.$$

These relations give us the formal solution to the wave equation with initial and boundary conditions.

Method of separation of variables for inhomogeneous PDEs.

Let us start studying inhomogeneous linear PDEs:

Inhomogeneous heat equation:

$$\begin{cases} u_t - K u_{xx} = h(x,t), & (x,t) \in [0,L] \times \mathbb{R}^+; \\ u(0,t) = u(L,t) = 0 & \text{Dirichlet boundary conditions} \\ u(x,0) = f(x) & \text{initial condition} \end{cases}$$

We now use what we already discovered for the homogeneous heat equation with Dirichlet b.c:

The admissible solutions are:

$$X_n = \alpha_n \sin\left(\frac{n\pi}{L}x\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n \in \mathbb{N}.$$

Now, instead of solving the ODE for $T(t)$ as in the homogeneous case we consider the general formula:

$$u(x,t) = \sum_{n=1}^{+\infty} T_n(t) \sin\left(\frac{n\pi}{L}x\right)$$

where T_n is arbitrary. By plugging the latter expression in the homogeneous PDE we obtain:

$$u_t - k u_{xx} = \sum_{n=1}^{+\infty} \left[T_n'(t) + k\left(\frac{n\pi}{L}\right)^2 T_n(t)\right] \sin\left(\frac{n\pi}{L}x\right).$$

(Rmk $T_n'(t) = (T_n)_t(t)$ but it's more compact). We now are left with the problem of finding T_n. Assume that, for every $t \in \mathbb{R}^+$, $C_n(t)$ is th n-th Fourier coefficient of the inhomogeneity $h(\cdot,t)$:

$$C_n(t) = \frac{2}{L}\int_0^L h(x,t)\left(\frac{n\pi}{L}x\right)dx.$$

Then we can express $h(x,t)$ as follows:

$$h(x,t) = \sum_{n=1}^{+\infty} C_n(t) \sin\left(\frac{n\pi}{L}x\right).$$

Then, the equation $u_t - k u_{xx} = h$ is equivalent to:

$$\sum_{n=1}^{+\infty} \left[T_n'(t) + k\left(\frac{n\pi}{L}\right)^2 T_n(t)\right] \sin\left(\frac{n\pi}{L}x\right) = \sum_{n=1}^{+\infty} C_n(t) \sin\left(\frac{n\pi}{L}x\right)$$

Therefore we need to impose:

$$T_n'(t) + k\left(\frac{n\pi}{L}\right)^2 T_n(t) = C_n(t)$$

and, to find the initial condition for T we use that

$$f(x) = u(x,0) = \sum_{n=1}^{+\infty} T_n(0)\, \sin\left(\frac{n\pi}{L}x\right).$$

Thus we have the following ODE system:

$$\begin{cases} T_n'(t) + k\left(\frac{n\pi}{L}\right)^2 T_n(t) = C_n(t) \\ T_n(0) = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi}{L}x\right) f(x)\, dx. \end{cases}$$

Thanks to the local existence and uniqueness thm for ODEs we have the existence of a unique solution $T_n(t)$ for every $n \in \mathbb{N}$. Then, the formal solution of our inhomogeneous Cauchy problem is:

$$u(x,t) = \sum_{n=1}^{+\infty} T_n(t)\, \sin\left(\frac{n\pi}{L}x\right).$$

Remark: as for the homogeneous case, if the boundary conditions were of Neumann's type, we would have an expansion in term of cosines, and there would be a summand for $n=0$.

Also, in the case of the inhomogeneous wave equation we would have a second order ODE for T_n, complemented with two initial conditions (for $T_n(0)$ and $T_n'(0)$) that will be linked to the Fourier expansion of $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ respectively.

Example (Section 5.4 PR)

Consider the inhomogeneous wave equation with Neumann boundary conditions:

$$\begin{cases} u_{tt} - u_{xx} = \cos(2\pi x)\cos(2\pi t) & \text{on } (0,1)\times\mathbb{R}^+ \\ u_x(0,t) = u_x(1,t) = 0 \\ u(x,0) = \cos^2(\pi x) \\ u_t(x,0) = 2\cos(2\pi x) \end{cases}$$

From the study of the method of separation of variables for the homogeneous wave equation with Neumann boundary conditions we have:

$$\begin{cases} X_{xx}(x) = -\lambda X(x) \quad \text{on } [0,1] \\ X_x(0) = X_x(1) = 0 \end{cases} \Rightarrow \begin{cases} X_n(x) = \cos(n\pi x) \\ \lambda_n = (n\pi)^2 \end{cases}, \quad n \geq 0$$

Note that at the moment we are not considering multiplicative factors in the expression for X_n because we still have to find the expression for T_n.

We are looking for solutions of the following form:

$$u(x,t) = \sum_{n=0}^{+\infty} T_n(t)\cos(n\pi x).$$

Imposing that $u(x,t)$ satisfies $u_{tt} - u_{xx} = \cos(2\pi x)\cos(2\pi t)$ we obtain:

$$u_{tt} - u_{xx} = \sum_{n=0}^{+\infty}\left[T_n''(t) + n^2\pi^2 T_n(t)\right]\cos(n\pi x) = \cos(2\pi x)\cos(2\pi t).$$

To find the $T_n(t)$ we consider the following cases:

$$T_n''(t) + n^2\pi^2 T_n(t) = 0 \quad \text{for } n \neq 2$$

$$T_2''(t) + 4\pi^2 T_2(t) = \cos(2\pi t) \quad \text{for } n = 2$$

Since $u(x,0) = \cos^2(\pi x) = \frac{1}{2} + \frac{1}{2}\cos(2\pi x)$ contains only summands of the form $\cos(n\pi x)$ for $n=0$ and $n=2$ we need to consider separately the case $n=0,2$. By the condition $u_t(x,0) = 2\cos(2\pi x)$ we get :

case $n=0$:
$$\begin{cases} T_0''(t) = 0 \\ T_0(0) = \frac{1}{2} \\ T_0'(0) = 0 \end{cases} \Rightarrow T_0(t) = \frac{1}{2}$$

case $n \geq 1, n \neq 2$
$$\begin{cases} T_n''(t) + n^2\pi^2 T_n(t) = 0 \\ T_n(0) = 0 \\ T_n'(0) = 0 \end{cases} \Rightarrow T_n(t) = 0$$

Case $n=2$
$$\begin{cases} T_2''(t) + 4\pi^2 T_2(t) = \cos(2\pi t) \qquad (**) \\ T_2'(0) = 2 \\ T_2(0) = \frac{1}{2} \end{cases}$$

$$\Rightarrow T_2(t) = \frac{1}{2}\cos(2\pi t) + \frac{1}{\pi}\sin(2\pi t) + \frac{t}{4\pi}\sin(2\pi t).$$

[Remark : the general solution of $(**)$ is the sum of the solution of tha associated homogeneous problem: $T_2''(t) + 4\pi^2 T_2(t) = 0$ and of a particular solution of the inhomogeneous one. However since $\cos(2\pi t)$ is both a solution of the homogeneous ODE and the inhomogeneity, we need to look for particular solutions of the form $t\sin(2\pi t)$ or $t\cos(2\pi t)$]

Finally, we obtained :

$$u(x,t) = \frac{1}{2} + \left[\frac{1}{2}\cos(2\pi t) + \frac{1}{\pi}\sin(2\pi t) + \frac{t}{4\pi}\sin(2\pi t)\right]\cdot\cos(2\pi x)$$

Example : (Inhomogeneous wave eq.)

$$\begin{cases} u_{tt} - u_{xx} = \sin(m\pi x)\sin(\omega t) & \text{on } (0,1)\times(0,+\infty) \\ u(0,t) = u(1,t) = 0 \rightsquigarrow \text{Dirichlet b.c.} & m \in \mathbb{N},\ \omega \in \mathbb{R} \\ \left.\begin{array}{l} u(x,0) = 0 \\ u_t(x,0) = 0 \end{array}\right\} \text{I.C.} \end{cases}$$

1. We consider solutions of the form $u(x,t) = X(x)T(t)$ as before. As for the homogeneous wave eq. with Dirichlet b.c. we get:

$$\begin{cases} X_{xx}(x) = -\lambda X(x) \\ X(0) = X(1) = 0 \end{cases} \Rightarrow X_n(x) = \sin(n\pi x),\quad \lambda_n = (n\pi)^2,\ n \geq 1$$

Now we look for $u(x,t) = \sum_{n=1}^{+\infty} T_n(t)\sin(n\pi x)$, and plugging it into the equation we have:

$$u_{tt} - u_{xx} = \sum_{n=1}^{+\infty}\left[T_n''(t) + n^2\pi^2 T_n(t)\right]\sin(n\pi x) = \sin(m\pi x)\sin(\omega t).$$

The ODEs for T_n are given by:

$$\begin{cases} T_n''(t) + n^2\pi^2 T_n(t) = 0 & \text{if } n \neq m \\ T_m''(t) + m^2\pi^2 T_m(t) = \sin(\omega t) \\ T_n(0) = 0 & \forall n \\ T_n'(0) = 0 & \forall n \end{cases}$$

Thus, for $n \neq m$ we have $T_n(t) = 0$. For $n = m$ we solve the inhomogeneous ODE $T_m''(t) + m^2\pi^2 T_m(t) = \sin(\omega t)$ and we get

$$T_m(t) = a_m\cos(m\pi t) + b_m\sin(m\pi t) + C_m\sin(\omega t).$$

(provided $\omega \neq m\pi$)

Using the initial conditions $T_m(0)=0$ and $T_m'(0)=0$ we obtain:

$$T_m(t) = \frac{1}{\omega^2 - m^2\pi^2}\left(\frac{\omega}{m\pi}\sin(m\pi t) - \sin(\omega t)\right).$$

The solution $u(x,t)$ is finally given by:

$$\boxed{u(x,t) = \frac{1}{\omega^2 - m^2\pi^2}\left(\frac{\omega}{m\pi}\sin(m\pi t) - \sin(\omega t)\right)\sin(m\pi x)}.$$

Remark: We are assuming $\omega \neq m\pi$ to avoid degeneracy.

To deal with the case $\omega = \pi m$ we can think to this case as limiting case as $\omega \neq \pi m$, $\omega \to m\pi$. Then,

$$\lim_{\omega \to m\pi} u(x,t) = \frac{1}{2m\pi}\left(\frac{\sin(m\pi t)}{m\pi} - t\cos(m\pi t)\right)\sin(m\pi x).$$

Note that for $\omega \neq \pi m$ $\forall m \in \mathbb{N}$, the solution is bounded. In other words, a bounded periodic force with time frequency ω different from the frequencies of the homogeneous solutions produces a bounded oscillations. On the other hand, for $\omega = m\pi$ for some $m \in \mathbb{N}$ the solution is unbounded. This is called resonance effect. You can look at the youtube video on the collapse of the Tacoma bridge.

The energy method and uniqueness

(Section 5.5)

One of the main applications of the energy method is to prove uniqueness of the solution of initial boundary value problems.

This method is based on the physical principle of conservation of energy, although the quantity we refer as "energy" may differ from the actual physical energy of the system. We illustrate the method in the following example:

Consider the inhomogeneous wave equation:

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) & \text{in } [0,L] \times \mathbb{R}^+ \\ u_x(0,t) = a(t) \\ u_x(L,t) = b(t) \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

Let u_1, u_2 be solutions and set $w := u_1 - u_2$. Then w solves:

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0 \\ w_x(0,t) = w_x(L,t) = 0 \\ w(x,0) = 0 \\ w_t(x,0) = 0 \end{cases}$$

Let us define the "energy" function:

$$E(t) = \int_0^L (w_t(t,x))^2 + c^2 (w_x(t,x))^2 \, dx .$$

By taking the derivative of E(t) we obtain: 9.12

$$\frac{d}{dt}E(t) = \int_0^L \Big(2 w_t w_{tt} + \underbrace{2c^2 w_x w_{xt}}_{\text{integrate by parts}}\Big)\,dx =$$

$$= 2\int_0^L \underbrace{\left(w_t w_{tt} - c^2 w_{xx} w_t\right)dx}_{= w_t\left[w_{tt} - c^2 w_{xx}\right] = 0} + \underbrace{\left[2c^2 w_x w_t\right]_0^L}_{w_x(0,t) = w_x(L,t) = 0} = 0$$

Therefore E(t) is constant, and since $E(0) = 0$ it follows that $E(t) = 0$ for all t.

By looking at the definition of E(t) we realize that $E(t) = 0 \;\; \forall t \Rightarrow w_x(x,t) = w_t(x,t) = 0 \;\; \forall t$.

Thus w is constant too. Because $w(x,0) = 0$ this implies $w(x,t) = 0 \;\; \forall x, t$. Thus $u_1 \equiv u_2$, which proves uniqueness.