Elliptic equations

- Today we will study the Laplace and the Poisson equations, that are the archetype of elliptic equations;
- We will learn the main properties of elliptic equations, the link between solutions of Laplace's eq. and harmonic functions.
- we will introduce the weak and strong maximum principle and the mean value theorem.

The Poisson equation $\Delta u = f$ and its homogeneous counterpart, the Laplace equation $\Delta u = 0$, have a very prominent role in applied sciences. For example, the temperature of an homogeneous and isotropic body at equilibrium is a solution of the Laplace equation. In this case we can say that the Laplace equation describes the stationary case (independent of time) of the diffusion equation.

Other examples are:

- the equilibrium position of a perfectly elastic membrane solves $\Delta u = 0$;
- the Poisson equation plays an essential role in the theory of conservative fields (electric field, magnetic field, gravitational field ...) If u is the electrostatic potential then the Poisson equation $\Delta u = f$ represents the link between the potential u and the charge density $-f$.

Basic properties of elliptic problems

(Section 7.2)

" on aura donc $\Delta u = 0$; cett équation remarquable nous sera de la plus grande utilité ... "
Laplace

We will study some basic models involving the Laplacian, including models for heat conduction, elasticity, electromagnetism, and gravitation.

We consider $u = u(x,y)$, $(x,y) \in D$ where D is an open set $D \subseteq \mathbb{R}^2$.

Definition: We say that u is harmonic if it solves the Laplace equation

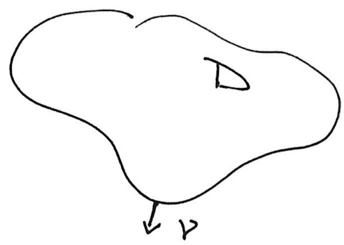
$$\Delta u(x,y) = u_{xx} + u_{yy} = 0. \quad (1)$$

The non-homogeneous version of the Laplace eq. is called Poisson equation: $\Delta u(x,y) = \rho(x,y)$. (2)

Remark: the Laplace and Poisson eq. are second order linear PDEs. The Laplace eq. is also homogeneous.

Remark: the linearity of the Laplace operator implies that a linear combination of harmonic functions is a harmonic function.

Definition: Let $D \subseteq \mathbb{R}^2$. Let ∂D be the boundary of D. Let ν be the unit outward normal to ∂D. Then we can consider the following Dirichlet problem:



$$(3) \begin{cases} \Delta u(x,y) = \rho(x,y), & (x,y) \in D \\ u(x,y) = g(x,y), & (x,y) \in \partial D \end{cases}$$

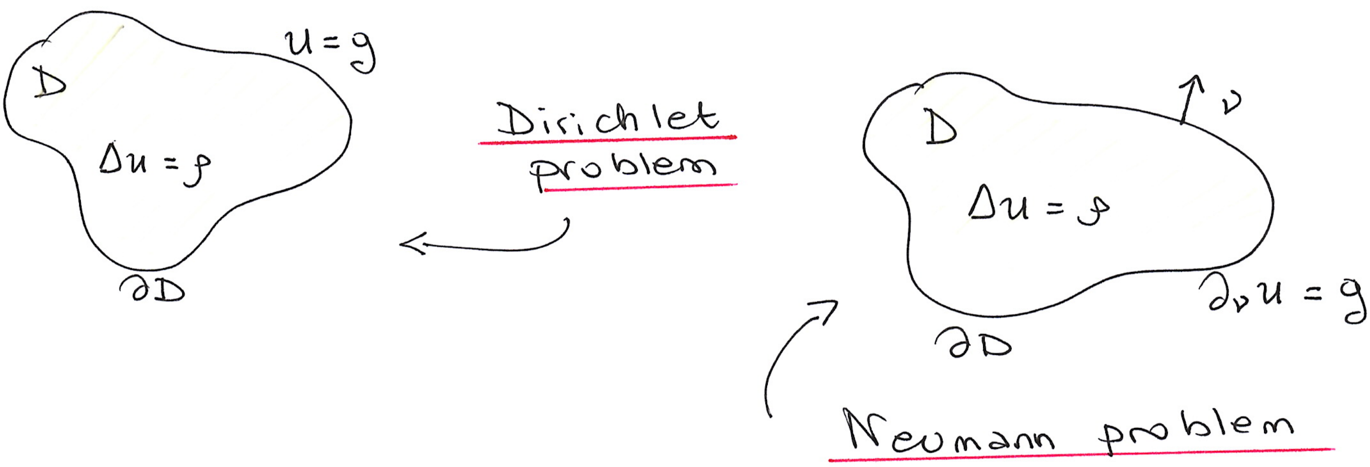
The Neumann problem for the Poisson eq. reads as follow:

$$(4)\begin{cases} \Delta u(x,y) = \rho(x,y), & (x,y)\in D \\ \partial_\nu u(x,y) = g(x,y), & (x,y)\in \partial D. \end{cases} \qquad (\partial_\nu u = \nu\cdot\nabla u)$$

Finally we can consider the problem of the third kind that is:

$$(5)\begin{cases} \Delta u(x,y) = \rho(x,y), & (x,y)\in D \\ u(x,y) + \alpha\, \partial_\nu u(x,y) = g(x,y), & (x,y)\in \partial D, \end{cases}$$

where α and g are given functions.



Question: do a solution to those problems exists? Consider the Neumann problem, this can model the distribution of the temperature $u(x,y)$ in the domain D at an equilibrium configuration. Then this means that the heat flux through the boundary must be balanced by the temperature production inside the domain. This simple consideration is encoded in the following lemma.

Lemma 7.4 A necessary condition for the existence of a solution to the Neumann problem (4) is :

$$\int_{\partial D} g(x(s), y(s))\, ds = \int_D \rho(x,y)\, dx\, dy \ , \quad \text{where}$$

$(x(s), y(s))$ is a parametrization of ∂D.

Proof : Recall the identity $\Delta u = \operatorname{div}(\nabla u)$. Then, the Poisson equation reads $\operatorname{div}(\nabla u) = \rho$.

If u is a solution of the Neumann problem we have :

$$\int_D \rho = \int_D \Delta u \overset{\Delta u = \operatorname{div}(\nabla u)}{=} \int_D \operatorname{div}(\nabla u) \underset{\text{Gauss' thm.}}{=} \int_{\partial D} \overbrace{\nabla u \cdot \nu}^{\text{definition of directional derivative}} = \int_{\partial D} \partial_\nu u =$$

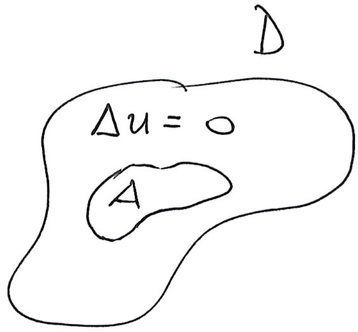
$$= \int_{\partial D} g .$$

Therefore $\int_D \rho = \int_{\partial D} g$ as desired. $\square$

Remark : If u is a solution of the Laplace equation $\Delta u = 0$, then we have that

$$\int_{\partial A} \partial_n u = \int_A \operatorname{div}(\nabla u) = 0$$

for every open subset $A \subseteq D$.



Question: Is the Cauchy problem for the Laplace equation well-posed (a solution exists, it is unique and it is stable with respect to the initial condition)? Recall: Cauchy problem:

$$\begin{cases} \Delta u = 0, & x \in \mathbb{R}, \ y > 0 \\ u(x,0) = f(x) \\ u_y(x,0) = g(x) \end{cases}$$

$\leadsto$ y plays the role of time (see wave equation)

Consider Laplace's equation in the half plane $x \in \mathbb{R}$, $y > 0$. The following counterexample to well-posedness is due to Hadamard:

Hadamard counterexample: Consider the following Cauchy problem:

$$\begin{cases} \Delta u(x,y) = 0, & x \in \mathbb{R}, \ y > 0 \\ u(x,0) = 0 & (*) \\ u_y(x,0) = \dfrac{\sin(nx)}{n} \quad \text{for } n \in \mathbb{N} & (**) \end{cases}$$

We look for solution of the form:

$$u(x,y) = \sin(nx) f(y).$$

Then, from $\Delta u(x,y) = 0$ we have:

$$0 = u_{xx} + u_{yy} = -n^2 \sin(nx) f(y) + \sin(nx) f''(y)$$

Therefore $\underline{f''(y) = n^2 f(y).}$

From the Dirichlet condition $u(x,0) = 0$ it follows that $\underline{f(0) = 0}$.

By the Neumann condition $u_y(x,0) = \dfrac{\sin(nx)}{n}$ we have:

$$u_y(x,0) = \frac{\sin(nx)}{n} = \sin(nx) f'(0) \implies \underline{f'(0) = \frac{1}{n}}.$$

Hence, solving $f''(y) = n^2 f(y)$ we obtain

$f(y) = \frac{1}{n^2}\sinh(ny)$, therefore

$$f(y) = A\sinh(ny) + B\cosh(ny)$$

$f(0) = 0 \Rightarrow B = 0$

$f'(0) = \frac{1}{n} \Rightarrow A = \frac{1}{n^2}$

$\sinh(0) = 0$

$\cosh(0) = 1$

$$\underline{u(x,y)} = \frac{1}{n^2}\underline{\sin(nx)\sinh(ny)}.$$

Setting $u^n(x,y) = \frac{1}{n^2}\sin(nx)\sinh(ny)$ we realise that in the limit $n \to +\infty$ both $u^n(x,0)$ and $u^n_y(x,0)$ tend to zero (the initial condition describes an arbitrarily small perturbation of the trivial solution $u=0$).

On the other hand, the solution is not bounded in the half plane $y>0$. Indeed, for any $a>0$

$$\sup_x |u^n(x,a)| = \sup_x \frac{1}{n^2}|\sin(nx)|\sinh(na) = \frac{1}{n^2}\sinh(na)$$

$$= \frac{1}{2n^2}\left(e^{na} - e^{-na}\right) \xrightarrow[n\to+\infty]{} +\infty.$$

Thus, the Cauchy problem for the Laplace equation is not stable and this implies that it is not well posed with respect to the initial conditions (*) and (**).

In the next example we will construct an initial datum for which there is no solution to the Cauchy problem. We will construct such initial datum using the Hadamard counterexample.

Consider as before $u^n(x,y) = \frac{1}{n^2}\sin(nx)\sinh(ny)$.

Then let's define:

$$\begin{cases} \overline{u}^N(x,y) = \sum_{n=1}^{N} \frac{u^n(x,y)}{n} \\ \overline{u}^N(x,0) = 0 \\ \overline{u}^N_y(x,0) = \sum_{n=1}^{N} \frac{u^n_y(x,0)}{n} = \sum_{n=1}^{N} \frac{1}{n^2}\sin(nx) \end{cases}$$

$\bar{u}^N$ is also a solution of the Laplace equation by linearity. However, for N that goes at infinity we don't have existence to the Cauchy problem

$$\begin{cases} \Delta u(x,y) = 0 & , \ x \in \mathbb{R}, y > 0 \\ u(x,0) = 0 \\ u_y(x,0) = \sum_{n=1}^{+\infty} \frac{1}{n^2} \sin(nx) \end{cases}$$

because the solution would be given by

$\bar{u}^{+\infty} := \sum_{n=1}^{+\infty} \frac{u^n(x,y)}{n}$ which is not a convergent serie. Note that the initial conditions $\bar{u}^{+\infty}(x,0)=0$ and $\bar{u}_y^{+\infty}(x,0) = \sum_{n=1}^{+\infty} \frac{1}{n^2} \sin(nx)$ do make perfectly sense.

Remark: These examples demonstrate the difference between elliptic and hyperbolic problems on the upper half plane.

Let us now compute some harmonic functions. We define harmonic polynomial od degree n to be a harmonic function $P_n(x,y)$ of the form:

$$P_n(x,y) = \sum_{0 \leq i+j \leq n} a_{ij} x^i y^j .$$

For example:

$n=0$: $u(x,y) = 1$

$n=1$: $u(x,y) = x$, $u(x,y) = y \longrightarrow u(x,y) = ax + by \quad \forall a,b \in \mathbb{R}$

$n=2$: $u(x,y) = xy$, $u(x,y) = x^2 - y^2$

$n=3$: $u(x,y) = x^3 - 3xy^2$, $u(x,y) = y^3 - 3x^2y$ ...

Polar coordinates

It can be very useful in applications, when for example the domain D has some radial symmetry, to express the Laplace operator in polar coordinates. We denote the polar coordinates (r, ϑ) with

$$\begin{cases} x = r\cos\vartheta \\ y = r\sin\vartheta \end{cases}$$

and the harmonic function by $w(r,\vartheta) = u(x(r,\vartheta), y(r,\vartheta))$.

Then the Laplacian in polar coordinates reads

$$\Delta u = w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\vartheta\vartheta} = 0.$$

Assume that $w = w(r)$, then $\quad 0 = w''(r) + \frac{1}{r} w'(r)$.
By substituting $v(r) = w'(r)$, we get $v'(r) = -\frac{v(r)}{r}$,
hence $\quad \frac{v'(r)}{v(r)} = -\frac{1}{r} \iff \frac{d}{dr} \log|v(r)| = -\frac{d}{dr} \log(r)$.

Thus, $\log|v(r)| = \underbrace{-\log(r)}_{= \log\left(\frac{1}{r}\right)} + c \implies w'(r) = v(r) = \frac{e^c}{r}$ if $v(r) > 0$, and $w'(r) = v(r) = -\frac{e^c}{r}$ if $v(r) < 0$. Then integrating with respect to r:

$$w(r) = \pm \int_1^r \frac{e^c}{s} \, ds \pm w(1) = c_1 \log(r) + c_2$$

(with $c_1 = e^c > 0$ if $v(r) > 0$ and $c_1 = -e^c > 0$ if $v(r) < 0$). Then $w(r) = c_1 \log(r) + c_2$ is a solution of the Laplace equation for $r > 0$. Since $r = \sqrt{x^2+y^2}$ this proves that $u(x,y) = w(r) = c_1 \log\left(\sqrt{x^2+y^2}\right) + c_2$
$= \frac{c_1}{2} \log(x^2+y^2) + c_2$, is harmonic on $\mathbb{R}^2 \setminus (0,0)$, $c_1, c_2 \in \mathbb{R}$.

Maximum principles

The maximum principle is a fundamental property of solutions to certain PDEs of elliptic or parabolic type. Maximum principles are based on the observation that if a C^2 function u attains its maximum over an open set D at a point $x_0 \in D$ then:

$$Du(x_0) = 0 \quad , \quad D^2 u(x_0) \leq 0,$$

where $D^2 u$ is the Hessian matrix. To use this observation we will need to work with solutions that are at least C^2.

Weak maximum principle: first we identify circumstances under which a function must attain its maximum (or minimum) on the boundary.

Theorem (weak maximum principle, thm 7.5) Let D be a bounded domain, and let $u(x,y) \in C^2(D) \cap C(\bar{D})$ be a harmonic function in D. Then the maximum of u in $\bar{D}$ is achieved on the boundary ∂D:

$$\max_{\bar{D}} u = \max_{\partial D} u$$

Proof: Consider the function $u_\varepsilon(x,y) = u(x,y) + \varepsilon(x^2+y^2)$, with $\varepsilon > 0$. Assume by contradiction that u_ε attains a local maximum at $(\bar{x},\bar{y}) \in D$. Then, $\Delta u_\varepsilon(\bar{x},\bar{y}) \leq 0$. On the other hand, since u is harmonic we have

$$\Delta u_\varepsilon(\bar{x},\bar{y}) = \Delta u(\bar{x},\bar{y}) + 4\varepsilon = 4\varepsilon > 0,$$ which is a contradiction.

This proves that u_ε takes its maximum on the boundary

$$\max_{\bar{D}} u_\varepsilon = \max_{\partial D} u_\varepsilon .$$

Thus, since $u \le u_\varepsilon$ and D is bounded:

$$\max_{\overline{D}} u \le \max_{\overline{D}} u_\varepsilon = \max_{\partial D} u_\varepsilon = \max_{\partial D} \left(u + \varepsilon(x^2+y^2)\right)$$

$$\le \max_{\partial D} u + \varepsilon \max_{\partial D} (x^2+y^2) = \max_{\partial D} u + \varepsilon C.$$

For $\varepsilon \to 0$ it follows that $\max_{\overline{D}} u \le \max_{\partial D} u$. $\square$

Corollary: Under the same hypotheses of the thm.

$$\min_{\overline{D}} u = \min_{\partial D} u.$$

Proof: Using that $\Delta(-u) = -\Delta u = 0$ it follows that

$$\min_{\overline{D}} u = -\max_{\overline{D}} (-u) = -\max_{\partial D} (-u) = \min_{\partial D} u. \quad \square$$

Remark: The boundedness of D is necessary. In fact, if one takes $D = \mathbb{R}^2 \setminus B_1$, then $u(x,y) = \log(x^2+y^2)$ is harmonic in D, $u|_{\partial D} = 0$ but $\sup_D u = +\infty \neq 0$.

Theorem (The mean value principle, thm 7.7) Consider u harmonic in D, let $B_R(x_0,y_0) \subseteq D$ be a ball of radius R. Then

$$u(x_0,y_0) = \frac{1}{2\pi R} \int_{\partial B_R(x_0,y_0)} u(x(s),y(s))\,ds = \frac{1}{2\pi} \int_0^{2\pi} u(x_0+R\cos\vartheta, y_0+R\sin\vartheta)\,d\vartheta$$

(7.14, PR)

Proof. Let $r \in (0,R)$, set

$$V(r) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0+r\cos\vartheta, y_0+r\sin\vartheta)\,d\vartheta.$$

And compute

$$V'(r) = \frac{1}{2\pi}\int_0^{2\pi} \frac{d}{dr} u(x_0 + r\cos\vartheta, y_0 + r\sin\vartheta)\, d\vartheta =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\vartheta \left(u_x(x_0 + r\cos\vartheta, y_0 + r\sin\vartheta)\cos\vartheta + u_y(x_0 + r\cos\vartheta, y_0 + r\sin\vartheta)\sin\vartheta \right)$$

$$= \frac{1}{2\pi r}\int_{\partial B_r(x_0,y_0)} \partial_\nu u = 0$$

here we use the remark after Lemma 7.4 : if $A \subseteq D$, $\int_{\partial A} \partial_\nu u = 0$.

see Remark at page 10.12

Thus $V(R) = V(0) \iff \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + R\cos\vartheta, y_0 + R\sin\vartheta)\, d\vartheta =$

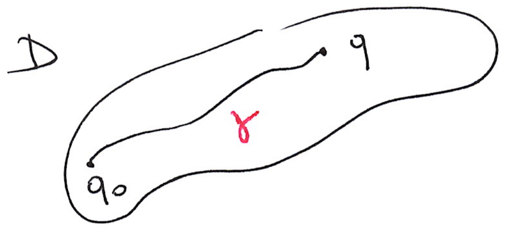
$= u(x_0, y_0)$

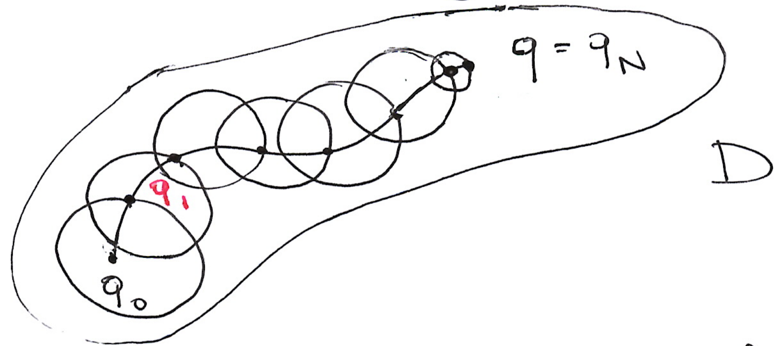
□

Remark : the inverse implication is also true : a smooth function that satisfies the mean value property in some domain D is harmonic in D.

Theorem (The strong maximum principle, thm 7.10)
Let u be harmonic in D, D connected. If u attains it maximum (minimum) at an interior point of D, then u is constant.

Proof : let $q_0 \in D$ be a maximum point. Let q be another point connected to q_0 by a curve γ.





then choose $R > 0$ smaller than the distance from γ to ∂D and define inductively a sequence of points $\{q_i\}_{i=0}^{N} \subset \gamma$ and radii $R_i < R$ such that $q_{i+1} \in \partial B_{R_i}(q_i)$

for any $i = 1, \ldots, N-1$, and $q_N = q$.

(Note that one can take $R_i = R$ for each $i = 0, \ldots, N-2$, and then $R_{N-1} \leq R$ such that $q_N \in \partial B_{R_{N-1}}(q_{N-1})$.)

Then inside each ball we apply inductively the mean value theorem. More precisely, by the mean value thm. applied at q_0 we have:

$$\max_D u = u(q_0) = \frac{1}{2\pi R} \int_{\partial B_R(q_0)} u \leq \frac{1}{2\pi R} \int_{\partial B_R(q_0)} \max_D u = \max_D u.$$

This implies that $u = \max_D u$ on $\partial B_R(q_0)$. Therefore since $q_1 \in \partial B_R(q_0)$ also q_1 is a point of maximum for u. Hence we can repeat the argument above (using the mean value thm.) to deduce that $u = \max_D u$ on $\partial B_R(q_1)$, hence $q_2 \in \partial B_R(q_1)$ is a maximum for u, and iterating we get that $q = q_N$ is a maximum for u. In particular

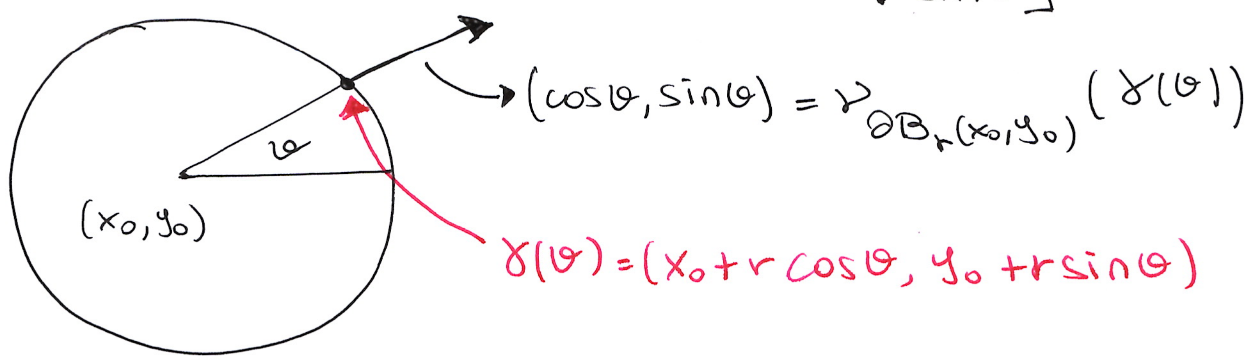
$$u(q_0) = \max_D u = u(q).$$

Since q is arbitrary this proves that $u = \max_D u = \text{constant}$ in D.

$\square$

Remark:

$$\frac{1}{2\pi} \int_0^{2\pi} d\theta \Big[u_x(\underbrace{x_0 + r\cos\theta, y_0 + r\sin\theta}_{\gamma(\theta)}) \cos\theta + u_y(x_0 + r\cos\theta, y_0 + r\sin\theta) \cdot \sin\theta \Big]$$



$\rightsquigarrow$ Our expression can be rewritten as

$$\frac{1}{2\pi}\int_0^{2\pi} \nabla u(\gamma(\vartheta)) \cdot \nu_{\partial B_r(x_0,y_0)}(\gamma(\vartheta))\, d\vartheta .$$

Let us call $F(\gamma(\vartheta)) = \nabla u(\gamma(\vartheta)) \cdot \nu_{\partial B_r}(\gamma(\vartheta))$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} F(\gamma(\vartheta))\, d\vartheta = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|\gamma'(\vartheta)|} \cdot F(\gamma(\vartheta)) \cdot |\gamma'(\vartheta)|\, d\vartheta$$

$$\gamma'(\vartheta) = (-r\sin\vartheta, r\cos\vartheta) \Rightarrow |\gamma'(\vartheta)| = r.$$

Thus, $\displaystyle\frac{1}{2\pi}\int_0^{2\pi} F(\gamma(\vartheta))\, d\vartheta = \frac{1}{2\pi r}\int_0^{2\pi} F(\gamma(\vartheta))\, |\gamma'(\vartheta)|\, d\vartheta$

$$\underset{\uparrow}{=} \frac{1}{2\pi r}\int_\gamma F .$$

by definition of integral along a curve

Since $F = \partial_\nu u$ and $\gamma = \partial B_r(x_0,y_0)$ we get the result.