Lecture 11 2-12-2019

- In this lecture we will continue the study of elliptic equations (uniqueness for the Poisson equation with Dirichlet boundary conditions, separation of variables for the Laplace equation)

- We will also introduce the maximum principle for the heat equation, and we also re-prove the uniqueness of solutions of the heat equation using the maximum principle.

Last week we proved some important results about harmonic functions (solutions of the Laplace eq. $\Delta u = 0$). In particular we proved the following results:

1. Weak maximum principle. Let D be a bounded domain (a domain is an open subspace of $\mathbb{R}^n$), and let $u(x,y) \in C^2(D) \cap C(\overline{D})$ be a harmonic function in D.

Then $$\max_{\overline{D}} u = \max_{\partial D} u$$

2. Mean value principle. Let u be harmonic in D, $B_R(x_0,y_0)$ be a ball of radius R centered at (x_0,y_0), $B_R(x_0,y_0) \subseteq D$. Then

$$u(x_0,y_0) = \frac{1}{2\pi R} \int_{\partial B_R(x_0,y_0)} u(x(s),y(s))\,ds$$

$$= \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + R\cos\vartheta,\, y_0 + R\sin\vartheta)\,d\vartheta$$

3. The strong maximum principle. Let u be harmonic in D, D connected domain. If u attains its maximum (minimum) at an interior point of D then u is constant. 11.2

Poisson equation (Section 7.4)

We will now see some important consequences of the maximum principles. Let us assume that the domain D is bounded, then we have the following:

Thm 7.12 (Uniqueness for the Dirichlet problem for the Poisson equation)
Consider the Dirichlet problem in a bounded domain

$$\begin{cases} \Delta u = f & \text{in } D \\ u = g & \text{in } \partial D \end{cases}$$

Then the problem has at most one solution $u \in C^2(D) \cap C(\bar{D})$.

Proof: Assume by contradiction that there exist two solutions u_1, u_2. Then define $u := u_1 - u_2$. This function u solves

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases} \leftarrow \text{by linearity}$$

From the weak maximum principle we get that the maximum and the minimum of u is zero, that implies $u \equiv 0 \implies u_1 = u_2$. $\square$

Remark: the domain D has to be bounded in order for thm 7.12 to hold. The reason for this is the same as for the weak maximum principle. Here is a counterexample: let u be a harmonic function in $\mathbb{R}^2 \setminus B_1(0)$, with $u \equiv 1$ on $\partial B_1(0)$. Then $u_1 \equiv 1$ is a solution to this problem, but also $u_2 = 1 + \log(x^2+y^2)$ is a solution $\Rightarrow$ no uniqueness.

Another consequence of the weak maximum principle.

Thm 7.13 (comparison between solutions of Dirichlet problems with different boundary conditions)

Let D be a bounded domain. Let $u_1, u_2 \in C^2(D) \cap C(\bar{D})$ solve $\Delta u_1 = 0$, $\Delta u_2 = 0$, with Dirichlet boundary conditions $u_1 \equiv g_1$ on ∂D and $u_2 \equiv g_2$ on ∂D.

Then
$$\max_{\bar{D}} |u_1 - u_2| = \max_{\partial D} |g_1 - g_2|.$$

Proof: Define $v = u_1 - u_2$. Then v is harmonic in D by linearity, and v satisfies $v = g_1 - g_2$ on ∂D. Therefore the maximum principle implies

$$\max_{\bar{D}} v = \max_{\partial D} v = \max_{\partial D} (g_1 - g_2),$$

and by the minimum principle (maximum principle for the minimum) we have

$$\min_{\bar{D}} v = \min_{\partial D} v = \min_{\partial D} (g_1 - g_2)$$

Therefore
$$\max_{\bar{D}} |u_1 - u_2| = \max_{\partial D} |g_1 - g_2|.$$
$\square$

Boundary conditions: we recall some different types of boundary conditions.

1. **Dirichlet** $u = g$ on ∂D

It may be referred also as condition of first type or as a fixed boundary condition. For example the following would be considered Dirichlet conditions:

a) In thermodynamics when a surface or an object is held at a fixed temperature;

b) In electrostatic when a node of a circuit is held at a fixed voltage;

c) In fluid dynamics, the no-slip condition for viscous fluids states that at a solid boundary the fluid will have zero velocity relative to the boundary.

2. **Neumann** $\partial_\nu u = g$ on ∂D, ν outer normal vector to D. This is also called second type boundary condition and it specifies the values in which the derivative of a solution is applied within the boundary of the domain.

Application: in thermodynamics a prescribed heat flux from a surface would serve as boundary condition. For example: a perfect insulator would have no flux, while an electrical component may be dissipating at a known power.

3. **Robin** or **third type boundary condition**:

$u + \alpha \partial_\nu u = g$ on ∂D, $\alpha \in \mathbb{R}$, g given function.

Robin boundary conditions are also called **impedance boundary conditions** from their application in elettro-magnetic problems.

Maximum principle for parabolic equations.

(Section 7.6)

In the previous lecture we studied the maximum principle for elliptic equations (in particular for the Laplace equation $\Delta u = 0$), but it also holds for parabolic equations.

Consider the heat equation for $u = u(t,x)$, $t > 0$, $x \in D$.

$$u_t = k \Delta u \qquad (1)$$

Define the domain

$Q_T = [0,T] \times D$, where D is the spatial domain and $t \in [0,T]$ is the time. Then we define the parabolic boundary $\partial_p Q_T = \{ \{0\} \times D \} \cup \{ [0,T] \times \partial D \}$, that is the boundary of Q_T except for the top cover $\{T\} \times D$.

Theorem 7.15 (Maximum principle for the heat equation)

Let u solve the homogeneous heat equation (1) in Q_T for some $k > 0$. Assume that D is bounded. Then u achieves its maximum (minimum) on $\partial_p Q_T$.

Proof: take $\varepsilon > 0$ and consider the function

$u_\varepsilon(t,x) = u(t,x) - \varepsilon t$. Then $\partial_t u_\varepsilon = \partial_t u - \varepsilon$, and

$\Delta u_\varepsilon = \Delta u$, therefore $\partial_t u_\varepsilon = k \Delta u_\varepsilon - \varepsilon$.

Assume by contradiction that u_ε has a maximum at some point $(t_0, x_0) \in Q_T \setminus \partial_p Q_T$. We distinguish two cases:

1. $t_0 < T$. In this case (t_0, x_0) is an interior max. point $\Rightarrow$ $\partial_t u_\varepsilon(t_0, x_0) = 0$, $\Delta u_\varepsilon(t_0, x_0) \leq 0$. This is in contradiction with the eq. $\partial_t u_\varepsilon = k \Delta u_\varepsilon - \varepsilon$.

2. $t_0 = T$. In this case (T, x_0) is an interior maximum point, therefore $\Delta u_\varepsilon(T, x_0) \leq 0$. On the other hand, since u_ε attains its maximum at (T, x_0):

$$\partial_t u_\varepsilon(T, x_0) = \lim_{s \to 0^+} \frac{u_\varepsilon(T, x_0) - u_\varepsilon(T-s, x_0)}{s} \geq 0.$$

Again these two inequalities are in contradiction with the equation $\partial_t u_\varepsilon = K \Delta u_\varepsilon - \varepsilon$.

In conclusion, u_ε attains its maximum on $\partial_p Q_T$.

Since $u - \varepsilon T \leq u_\varepsilon \leq u$ inside Q_T we get:

$$\max_{\partial_p Q_T} u \geq \max_{\partial_p Q_T} u_\varepsilon = \max_{\overline{Q_T}} u_\varepsilon \geq \max_{\overline{Q_T}} u - \varepsilon T$$

and the result follows letting $\varepsilon \to 0$. $\square$

Corollary. Consider the Dirichlet problem for the heat equation:

$$\begin{cases} u_t - K \Delta u = f & \text{in } Q_T \\ u(0,x) = g & \text{on } D \\ u(t,x) = h & \text{on } [0,T] \times \partial D \end{cases}$$

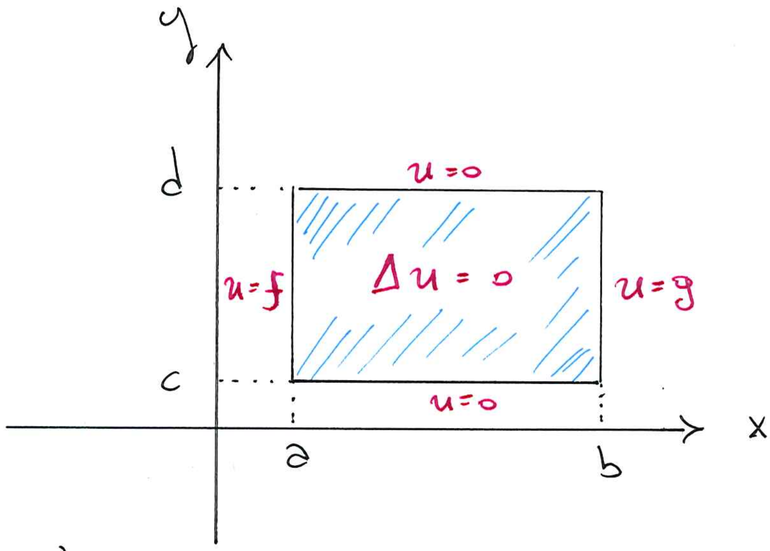
Then this problem has a unique solution.

Proof (EXERCISE). Consider the function $v = u_1 - u_2$ and look at the equation fulfilled by v, that is the homogeneous heat equation with zero boundary and initial conditions, leading to $v = 0 \Rightarrow u_1 = u_2$. $\square$

(see also thm 7.17)

Separation of variables

We will now use the method of separation of variables to solve the Laplace eq. on rectangular domains.



Assume $R = [a,b] \times [c,d]$, we need to assign one boundary condition on every side of the rectangle. This is what is a Dirichlet boundary condition (the solution u is required to satisfy different Dirichlet conditions on disjoints parts of the boundary of the domain).

As a first example we start with the assumption that $u=0$ on two opposite sides of the rectangle:

$$\begin{cases} \Delta u = 0 & \text{in } R \\ u = 0 & \text{on } [a,b] \times \{c,d\} \\ u = f & \text{on } \{a\} \times [c,d] \\ u = g & \text{on } \{b\} \times [c,d] \end{cases}$$

We look for a solution of the form: $u(x,y) = X(x)\,Y(y)$. By plugging it in the equation we get $X''(x)\,Y(y) + Y''(y)\,X(x) = 0$. Dividing by $X(x)Y(y)$ we obtain:

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}.$$

Since the function on the left only depends on x, while the one on the right only depends on y, the only possibility is that they are both constant:

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda \in \mathbb{R}.$$

For Y we have the following ODE: $Y''(y) = -\lambda Y(y)$ and by the analysis we did in the previous lectures we know that equations of this type have three solutions depending on the sign of λ. By the condition $Y(c) = Y(d) = 0$ we deduce that λ must be positive:

$$\lambda = \lambda_n = \left(\frac{n\pi}{d-c}\right)^2, \quad n \in \mathbb{N},\ n \geq 1.$$

and

$$Y_n(y) = a_n \sin\left(\frac{n\pi(y-c)}{d-c}\right), \quad n \in \mathbb{N},\ n \geq 1.$$

Important remark: If we had Neumann boundary conditions for Y, then we would have used a cosines expansion instead of a sines one and we would have started at $n=0$.

Concerning the ODE for X we have: $X_n''(x) = \lambda_n X_n(x)$. Since $\lambda_n > 0$ $\forall n \geq 1$, the solution of our sequence of ODEs is a combination of $\sinh(\sqrt{\lambda_n}\, x)$ and $\cosh(\sqrt{\lambda_n}\, x)$. However, instead of expressing the family of all solutions to the ODE as linear combination of sinh and cosh in x, because of our boundary conditions, we will express our solution in terms of

$$\sinh(\sqrt{\lambda_n}\,(x-a)), \quad \sinh(\sqrt{\lambda_n}\,(x-b)), \quad \lambda_n = \left(\frac{n\pi}{d-c}\right)^2$$

Therefore the general form of X_n is given by:

$$X_n(x) = \alpha_n \sinh\left(\sqrt{\lambda_n}\,(x-a)\right) + \beta_n \sinh\left(\sqrt{\lambda_n}\,(x-b)\right).$$

Altogether, the expressions for X_n and Y_n give:

$$u(x,y) = \sum_{n=1}^{+\infty} \left[A_n \sinh\left(\sqrt{\lambda_n}\,(x-a)\right) + B_n \sinh\left(\sqrt{\lambda_n}\,(x-b)\right)\right] \sin\left(\sqrt{\lambda_n}\,(y-c)\right)$$

where we renamed the coefficients. Now the only task left is to determine the coefficients A_n and B_n.

To do so we can use the boundary conditions.

Taking $x = a$, since $\sinh(0) = 0$ we get:

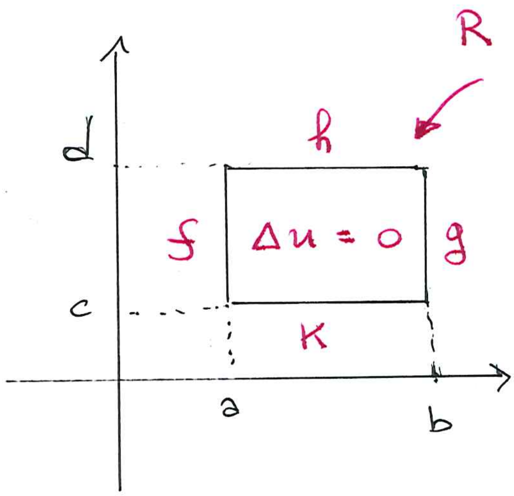
$$u(a,y) = \sum_{n=1}^{+\infty} B_n \sinh\left(\sqrt{\lambda_n}\,(a-b)\right) \sin\left(\sqrt{\lambda_n}\,(y-c)\right) = f(y),$$

from which we deduce that B_n are the Fourier coefficients of f, scaled by a factor $\sinh\left(\sqrt{\lambda_n}\,(a-b)\right)$. The same reasoning applies to the other boundary condition in order to determine A_n.

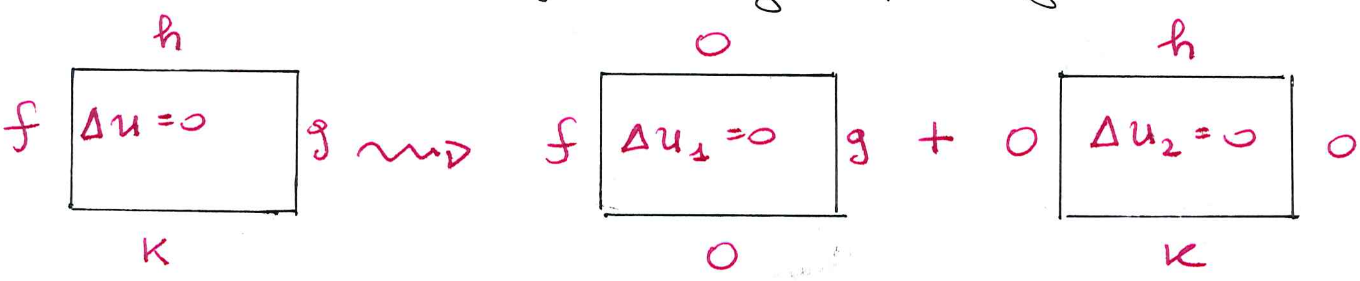
Boundary splitting In the example above we had two opposite boundaries where u was zero. This simplified our lives and allowed us to have an expansion in sines for Y and $\sinh$ for X.

Let us now consider the general case where we have nonzero boundary conditions:

$$\begin{cases} \Delta u = 0 & \text{in } R \\ u = f & \text{on } \{a\} \times [c,d] \\ u = g & \text{on } \{b\} \times [c,d] \\ u = h & \text{on } [a,b] \times \{d\} \\ u = k & \text{on } [a,b] \times \{c\} \end{cases}$$



Then we can do the following splitting:



$u = u_1 + u_2$

Splitting of the boundary conditions

We use the linearity.

We consider $u = u_1 + u_2$ where:

u_1 solves:

$$\begin{cases} \Delta u_1 = 0 & \text{in } R \\ u_1 = f & \text{on } \{a\}\times[c,d] \\ u_1 = g & \text{on } \{b\}\times[c,d] \\ u_1 = 0 & \text{on } [a,b]\times\{d\} \\ u_1 = 0 & \text{on } [a,b]\times\{c\} \end{cases}$$

u_2 solves:

$$\begin{cases} \Delta u_2 = 0 & \text{in } R \\ u_2 = 0 & \text{on } \{a\}\times[c,d] \\ u_2 = 0 & \text{on } \{b\}\times[c,d] \\ u_2 = h & \text{on } [a,b]\times\{d\} \\ u_2 = k & \text{on } [a,b]\times\{c\} \end{cases}$$

We have seen in the previous example that u_1 is of the form

$$u_1(x,y) = \sum_{n=1}^{+\infty}\left[A_n \sinh\left(\frac{n\pi}{d-c}(x-a)\right) + B_n \sinh\left(\frac{n\pi}{d-c}(x-b)\right)\cdot \sin\left(\frac{n\pi}{d-c}(y-c)\right)\right.$$

Analogously, reversing the role of x and y, u_2 will be given by an expression of the form:

$$u_2(x,y) = \sum_{n=1}^{+\infty}\left[C_n \sinh\left(\frac{n\pi}{b-a}(y-c)\right) + D_n \sinh\left(\frac{n\pi}{b-a}(y-d)\right)\cdot \sin\left(\frac{n\pi(x-a)}{b-a}\right)\right.$$

Then the coefficients A_n, B_n, C_n and D_n are related to the Fourier coefficients of the boundary date.

Remark: Notice that, when we split the problem for u into two problems for u_1 and u_2 the boundary date may not be continuous anymore even if they were continuous in the original problem. (consider for instance the case $f = g = h = k = 1$).

This is not an issue analytically, but it becomes a problem when one wants to solve the problem numerically, since the jump in the boundary data creates numerical problems.

A trick to avoid this is the following: if we want to solve

$$\begin{cases} \Delta u = 0 & \text{in } R \\ u = f & \text{on } \{a\} \times [c,d] \\ u = g & \text{on } \{b\} \times [c,d] \\ u = h & \text{on } [a,b] \times \{d\} \\ u = K & \text{on } [a,b] \times \{c\} \end{cases}$$

we can still define $\overline{u} := u - P$ with P a polynomial $P := a_0 + a_1 x + a_2 y + a_3 xy$ for some $a_0, a_1, a_2, a_3 \in \mathbb{R}$. Note that $\overline{u}$ is still harmonic since P is harmonic, and it solves:

$$\begin{cases} \Delta \overline{u} = 0 & \text{in } R \\ \overline{u} = \overline{f} & \text{on } \{a\} \times [c,d] \\ \overline{u} = \overline{g} & \text{on } \{b\} \times [c,d] \\ \overline{u} = \overline{h} & \text{on } [a,b] \times \{d\} \\ \overline{u} = \overline{K} & \text{on } [a,b] \times \{c\} \end{cases}$$

where $\overline{f} = f - P$, $\overline{g} = g - P$, $\overline{h} = h - P$, $\overline{K} = K - P$. Now, if the boundary data for u were continuous, we can choose the coefficients $a_0, a_1, a_2, a_3 \in \mathbb{R}$ to ensure that $\overline{f}(a,c) = \overline{f}(a,d) = 0$ and $\overline{g}(b,c) = \overline{g}(b,d) = 0$ (we have 4 parameters to adjust 4 boundary conditions). In this way if we consider:

$$\begin{cases} \Delta \overline{u}_1 = 0 & \text{in } R \\ \overline{u}_1 = \overline{f} & \text{on } \{a\} \times [c,d] \\ \overline{u}_1 = \overline{g} & \text{on } \{b\} \times [c,d] \\ \overline{u}_1 = 0 & \text{on } [a,b] \times \{d\} \\ \overline{u}_1 = 0 & \text{on } [a,b] \times \{c\} \end{cases} \quad \& \quad \begin{cases} \Delta \overline{u}_2 = 0 & \text{in } R \\ \overline{u}_2 = 0 & \text{on } \{a\} \times [c,d] \\ \overline{u}_2 = 0 & \text{on } \{b\} \times [c,d] \\ \overline{u}_2 = \overline{h} & \text{on } [a,b] \times \{d\} \\ \overline{u}_2 = \overline{K} & \text{on } [a,b] \times \{c\} \end{cases}$$

the boundary data for $\overline{u}_1$ and $\overline{u}_2$ are
not discontinuous anymore.