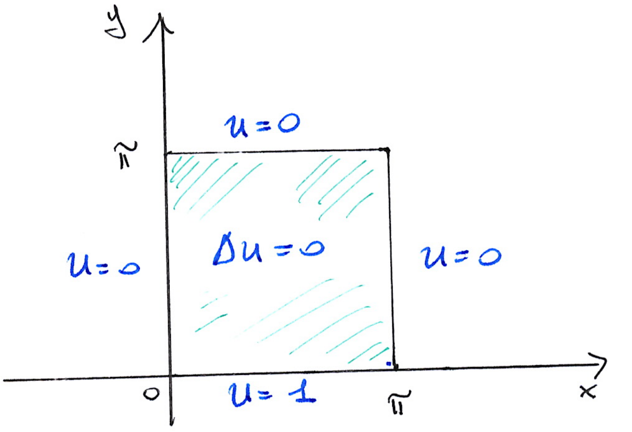
The method of separation of variables for elliptic equations on rectangular domains and on circular domains. (Sect. 7.7.1 - 7.7.2)

In last lecture we studied some consequences of maximum principles for elliptic equations, the maximum principle for the heat equation, and we solved the Laplace equation on rectangles using separation of variables.
Today we will see examples of solutions of the Laplace equation on rectangles with Dirichlet and Neumann boundary conditions.
Finally we will consider the Laplace equation on circular domains where we will make use of polar coordinates.

Example (Laplace eq. in the square with Dirichlet boundary conditions) We want to solve the following Dirichlet problem on a square $[0,\pi]\times[0,\pi] \subseteq \mathbb{R}^2$:

$$\begin{cases} \Delta u = 0 & \text{in } R=[0,\pi]\times[0,\pi]; \\ u(x,0)=1 \\ u(x,\pi)=u(0,y)=u(\pi,y)=0 \end{cases}$$



Since we only have one nonzero boundary condition, there is no need to split the problem.(*) We look for a solution of the form
$$u(x,y)=\sum_n X_n(x)\,Y_n(y)$$
where each term $X_n(x)\,Y_n(y)$ is harmonic. Hence,

$$0=\Delta\big(X_n(x)\,Y_n(y)\big)=X_n''(x)\,Y_n(y)+X_n(x)\,Y_n''(y) \iff$$

$$\frac{X_n''(x)}{X_n(x)}=-\frac{Y_n''(y)}{Y_n(y)}=-\lambda_n\in\mathbb{R}.$$

(*) see Lecture 11, page 11.9.

Therefore we get two ODEs:

$$\begin{cases} X_n''(x) = -\lambda_n X_n(x), & X_n(0) = X_n(\pi) = 0 \\ Y_n''(y) = \lambda_n Y_n(y) \end{cases}$$

From this we have the solution for X_n:

$$X_n(x) = A_n \operatorname{sen}(\sqrt{\lambda_n}\, x) + B_n \cos(\sqrt{\lambda_n}\, x)$$

From $X_n(0) = 0$ we have $B_n = 0 \ \forall n \in \mathbb{N}$.

$X_n(x) = A_n \operatorname{sen}(\sqrt{\lambda_n}\, x)$ and from $X_n(\pi) = 0$ we have

$$\operatorname{sen}(\sqrt{\lambda_n}\, \pi) = 0 \implies \lambda_n = \left(\frac{n\cancel{\pi}}{\cancel{\pi}}\right)^2 \implies \lambda_n = n^2 \ \forall n \in \mathbb{N},$$

$n \geq 1$.

Finally, $X_n(x) = A_n \sin(nx)$, $\lambda_n = n^2$. (Note that the coefficients A_n will be then "included" in the coefficients of the Fourier series).

The function Y_n is given by:

$$Y_n(y) = C_n \sinh(ny) + D_n \sinh(n(y-\pi)).$$

Recap on how the solution for Y_n is obtained:

$Y_n''(y) = n^2 Y_n(y)$. Solutions to this problems are:

$Y_n(y) = \alpha_n \sinh(ny) + \beta_n \cosh(ny)$, but in the case of rectangular domains with Dirichlet boundary conditions it is more convenient to express the solution in terms of $\sinh(n(y-0)) = \sinh(ny)$ and $\sinh(n(y-\pi))$.

To see in detail how to pass from the previous basis to the new one you can look at the solution of the Exercise 11.3 (a) of the Exercise sheet 11.

So now we have:

$X_n = A_n \sin(nx)$ and $Y_n = C_n \sinh(ny) + D_n \sinh(n(y-\pi))$.

Therefore the general solution is:

$$u(x,y) = \sum_n \sin(nx)\left[C_n \sinh(ny) + D_n \sinh(n(y-\pi))\right]. \quad (1)$$

Remark: note that the general form of the solution is given by:

$$u(x,y) = \sum_n X_n(x)\, Y_n(y) = \sum_n A_n \sin(nx)\left[C_n \sinh(ny) + D_n \sinh(n(y-\pi))\right].$$

Up to absorbing the constants A_n inside C_n and D_n we obtain formula (1).

From the condition $u(x,\pi) = 0$, we obtain $C_n = 0$ for all $n \in \mathbb{N}^+$. Then, from $u(x,0) = 1$ we have:

$$1 = u(x,0) = \sum_n \sin(nx)\left[D_n \sinh(-n\pi)\right].$$

To determine the coefficients D_n we introduce the coefficients $d_n = D_n \sinh(-n\pi)$ so that the condition above becomes:

$$1 = \sum_n d_n \sin(nx).$$

As usual we multiply both sides by $\sin(mx)$ and we integrate over $[0,\pi]$:

$$\int_0^\pi \sin(mx)\,dx = \sum_n d_n \int_0^\pi \sin(mx)\sin(nx)\,dx = d_m \int_0^\pi \sin^2(mx)\,dx$$

$$= d_m \frac{\pi}{2}$$

$$\left(\int_0^\pi \sin(mx)\sin(nx)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{\pi}{2} & \text{if } m = n \end{cases} \right)$$

Thus, $\alpha_m = \frac{2}{\pi} \int_0^{\pi} \sin(mx)\,dx = \frac{2}{\pi}\left(- \frac{\cos(mx)}{m} \Big|_0^{\pi} \right)$

$$= \frac{2}{\pi}\left[\frac{1-\cos(m\pi)}{m} \right] = \begin{cases} \frac{4}{\pi m}, & m = 2j+1 \\ 0, & m = 2j \end{cases}$$

Therefore, since $\alpha_m = D_m \sinh(-m\pi)$ we have:

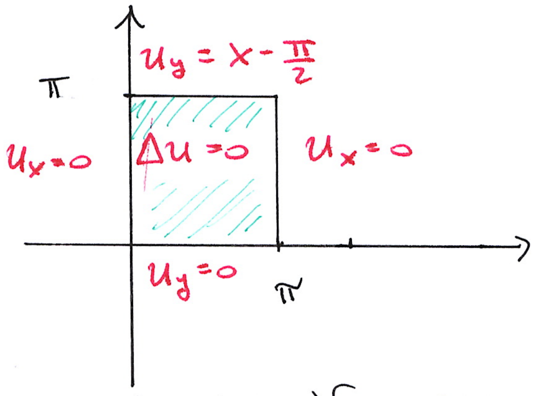
$$D_m = \begin{cases} \frac{4}{\pi m \sinh(-m\pi)}, & m = 2j+1 \\ 0, & m = 2j \end{cases}.$$

In conclusion, since $\sinh(-m\pi) = -\sinh(m\pi)$:

$$u(x,y) = \sum_{j=0}^{+\infty} \sin((2j+1)x) \cdot \left[\frac{-4 \sinh((2j+1)(y-\pi))}{\pi(2j+1)\sinh((2j+1)\pi)} \right].$$

Example Consider now the Laplace eq. on R as above with **Neumann** boundary conditions:

$$\begin{cases} \Delta u = 0 \quad \text{in } R = [0,\pi]\times[0,\pi] \\ u_y(x,\pi) = x - \frac{\pi}{2} \\ u_x(0,y) = u_x(\pi,y) = u_y(x,0) = 0 \end{cases}$$



Recall the necessary condition to solve elliptic Neumann problems:

$$\int_{\partial D} \partial_\nu u = \int_{\partial D} g = \int_D 0 = 0 \qquad \left(\begin{cases} \Delta u = 0 & \text{on } D \\ \partial_\nu u = g & \text{on } \partial D \end{cases} \right)$$

Therefore we need to check whether this condition is verified:

$$\int_{\partial D} \partial_\nu u = \int_0^{\pi} \left(x - \frac{\pi}{2}\right) dx = 0 \quad \checkmark$$

Since the necessary condition is verified we can 12.5
proceed in looking for a solution using the method of separation of variables:

$u(x,y) = \sum_n X_n(x) Y_n(y)$, $\Delta u = 0$ leads to

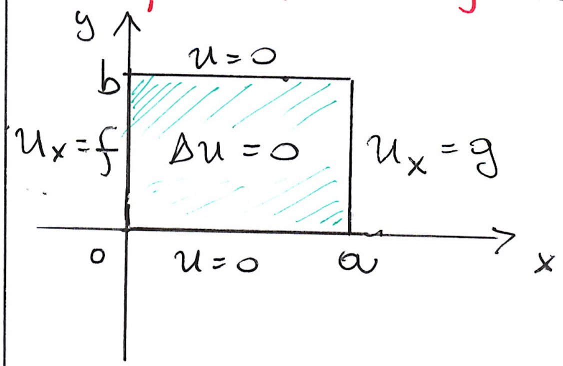
$$\begin{cases} X_n''(x) = -\lambda_n X_n(x), & X_n'(x) = X_n'(\pi) = 0 \\ Y_n''(y) = \lambda_n Y_n(y) \end{cases}$$

Since we have Neumann boundary conditions we obtain

$X_n(x) = \cos(nx)$ with $\lambda_n = n^2$, $n \geq 0$.

$Y_n(y) = A_n \cosh(ny) + B_n \cosh(n(y-\pi))$.

Remark: you can also have Dirichlet conditions on some parts of the boundary and Neumann conditions on other parts of the boundary and in this case you need to choose the right bases in terms of $\{\sin, \cos\}$ and $\{\sinh, \cosh\}$. For instance if you have:



Then $u(x,y) = \sum_n X_n(x) Y_n(y)$

$$u(x,y) = \sum \sin(\sqrt{\lambda_n}\, y)\left[A_n \cosh(\sqrt{\lambda_n}\, x) + B_n \cosh(\sqrt{\lambda_n}\,(x-a))\right]$$

$\lambda_n = \left(\frac{n\pi}{b}\right)^2$, $n \in \mathbb{N}$.

The general solution is:

$$u(x,y) = \sum_{n=0}^{+\infty} \cos(nx)\left[A_n \cosh(ny) + B_n \cosh(n(y-\pi))\right].$$

Using the boundary conditions to find A_n and B_n we have:

$$0 = u_y(x,0) = \sum_{n=0}^{+\infty} \cos(nx)\, B_n\, n \sinh(-n\pi) \Rightarrow B_n = 0,$$

and

$$x - \frac{\pi}{2} = u_y(x,\pi) = \sum_{n=0}^{+\infty} A_n n \sinh(n\pi)\cos(nx) = \sum_{n=0}^{+\infty} \beta_n \cos(nx),$$

where $\beta_n = A_n n \sinh(n\pi)$. By a similar computation as the one in the previous example we get (check it as an exercise)

$$\beta_m = \begin{cases} -\frac{4}{\pi m^2}, & m = 2j+1 \\ 0, & m = 2j \end{cases} \Rightarrow A_m = \begin{cases} \frac{-4}{\pi m^3 \sinh(m\pi)}, & m = 2j+1 \\ 0 & m = 2j \end{cases}$$

$m \geq 1$. Remark: $\beta_0 = 0$ independently of the value of A_0. This yields the solution:

$$u(x,y) = \sum_{j=0} \frac{-4\cos((2j+1)x)\cosh((2j+1)y)}{\pi\,(2j+1)^3 \sinh((2j+1)\pi)} + A_0.$$

Note that this is a solution for every value of A_0.

Splitting of the problem in case of Neumann boundary conditions. Consider now the Laplace eq. in a rectangular domain with Neumann boundary conditions:

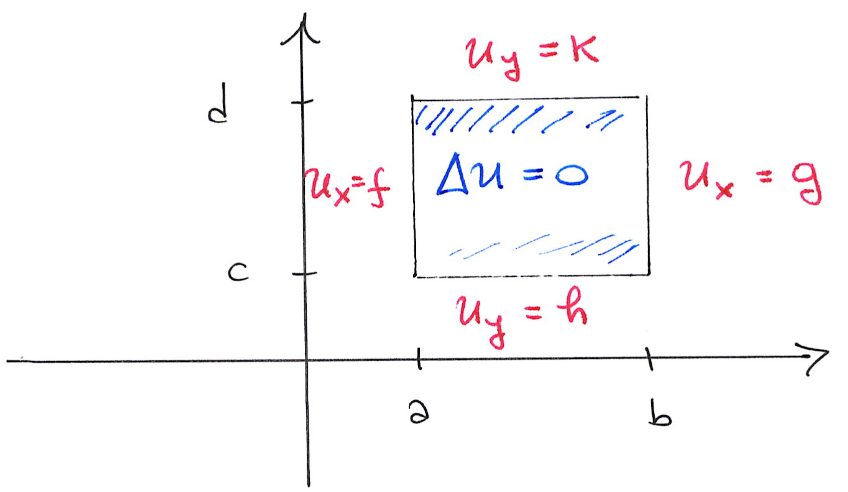
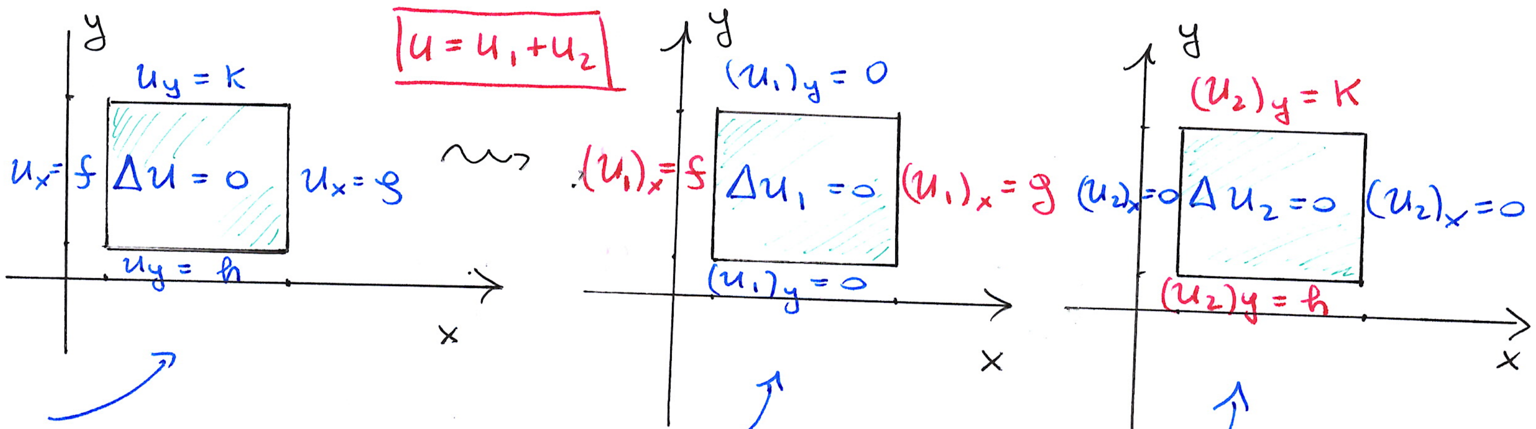


Fig 1

Remark: suppose that the problem in fig. 1 satisfies the necessary condition for the existence of a solution to the Neumann problem, namely

$$\int_c^d g - \int_c^d f + \int_a^b K - \int_a^b h = 0 .$$

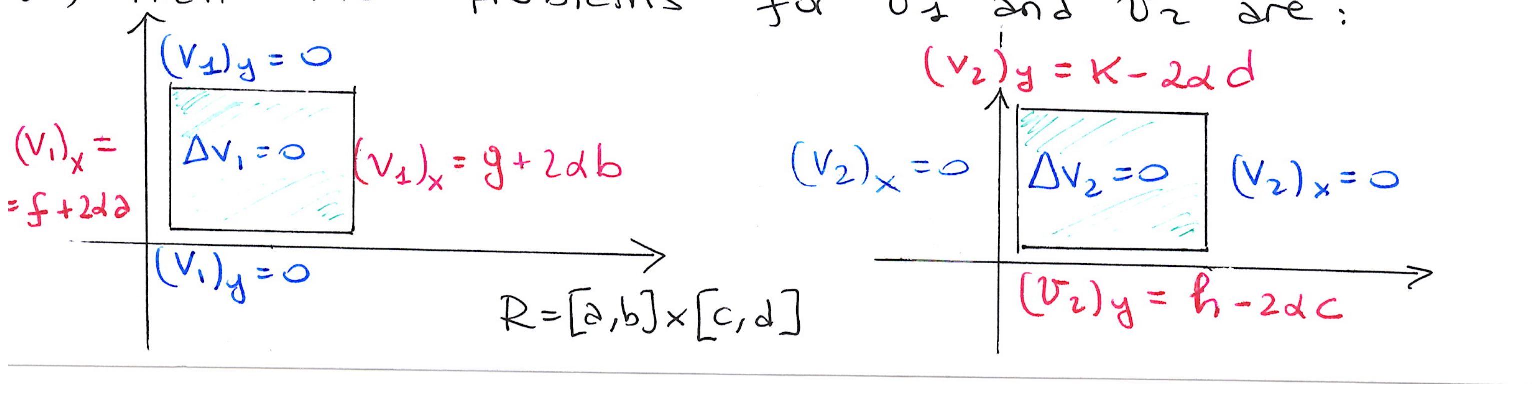
To solve the problem we need to split it in the sum of two problems as we did in lecture 11 for the Dirichlet problem.



Problem for u — Problem for u_1 — Problem for u_2

Warning: by splitting the domain the existence condition for the Neumann problem might not be satisfied any more.

To overcome this problem, we use the trick of adding a harmonic polynomial: consider for instance $\alpha(x^2 - y^2)$ with $\alpha \in \mathbb{R}$, and add it to u. This yields the new harmonic function $v = u + \alpha(x^2 - y^2)$. If we now split $v = v_1 + v_2$ (as we were doing above for u) then the problems for v_1 and v_2 are:



Note that the compatibility condition for v_1 is given by:

$$\int_c^d [g+2\alpha b] - \int_c^d [f+2\alpha a] = 0 \implies \alpha = \frac{1}{2(b-a)(d-c)}\left[\int_c^d (f-g)\right].$$

Hence, with this choice of α we can solve the problem for v_1. Recall now that, by assumption, the Neumann problem for u was solvable, that is

$$\int_c^d g - \int_c^d f + \int_a^b k - \int_a^b h = 0.$$

Thus α is also equal to $\frac{1}{2(b-a)(d-c)}\left[\int_a^b (k-h)\right]$.

This implies that

$$\int_a^b [k-2\alpha d] - \int_a^b [h-2\alpha c] = 0 \checkmark$$

therefore also v_2 satisfies the compatibility condition and we can solve the problem using the method of separation of variables.

The Laplace equation in circular domains. We conclude this lecture considering the Laplace eq. on circular domains $D = B_a = \{0 \le r < a, \, \theta \in [0, 2\pi]\}$. For this problem we use the equation in polar coordinates

$$0 = \Delta u = w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\theta\theta},$$

where $u(x(r,\theta), y(r,\theta)) = u(r\cos\theta, r\sin\theta) = w(r,\theta)$. We look for separated solutions of the form

$$w(r,\theta) = R(r)\Theta(\theta)$$

and obtain:

$$0 = R''(r)\Theta(\vartheta) + \frac{1}{r}R'(r)\Theta(\vartheta) + \frac{1}{r^2}\Theta''(\vartheta)R(r) \Rightarrow$$

$$\frac{r^2R''(r) + rR'(r)}{R(r)} = -\frac{\Theta''(\vartheta)}{\Theta(\vartheta)} = +\lambda,$$

for which the ODEs system is:

$$\begin{cases} r^2R''(r) + rR'(r) = \lambda R(r) \\ \Theta''(\vartheta) = -\lambda\Theta(\vartheta), \quad \Theta(0) = \Theta(2\pi), \quad \Theta'(0) = \Theta'(2\pi). \end{cases}$$

Note that the conditions $\Theta(0) = \Theta(2\pi)$, $\Theta'(0) = \Theta'(2\pi)$ come from the fact that we want u to be a classical solution inside D, so it should be at least C^2. Hence we impose that Θ and Θ' should be periodic in $[0, 2\pi]$. Observe that, since $\Theta'' = -\lambda\Theta$ automatically also Θ'' is periodic. The solution for the second ODE is:

$\Theta_n(\vartheta) = A_n\cos(n\vartheta) + B_n\sin(n\vartheta)$, for $\lambda_n = n^2$. For the first equation one can check that

$$R_n(r) = \begin{cases} C_0 + D_0\log(r), & n = 0 \\ C_n r^n + D_n r^{-n}, & n \neq 0 \end{cases}$$

gives the two parameter family of solutions (recall that linear second order ODEs have always a two parameter family of solutions). However the functions r^{-n} and $\log(r)$ are singular at 0 inside the domain D, so we discard them. Thus the general solution is given by:

$$w(r,\vartheta) = C_0 + \sum_{n=1}^{+\infty} r^n\left[A_n\cos(n\vartheta) + B_n\sin(n\vartheta)\right].$$