Recap of lecture 12. In the last lecture(s) we studied the Laplace equation: from the maximum principle to the solution of the Dirichlet and Neumann problems using the method of separation of variables.

We studied these problems on rectangular domains and on the disc. Let me remind you now the method of separation of variables works if we use polar coordinates.

Consider a function $u = u(x,y) \in \mathbb{R}^2$ and, define

$$w(r,\vartheta) := u(r\cos\vartheta, r\sin\vartheta), \quad r \geq 0, \quad \vartheta \in [0,2\pi).$$

$$\begin{cases} x = r\cos\vartheta \\ y = r\sin\vartheta \end{cases}, \quad r = \sqrt{x^2+y^2}, \quad \vartheta = \arctan\left(\frac{y}{x}\right).$$

We want to solve the Laplace equation in a domain with radial symmetry like, for example, the disc $D_a := \{0 \leq r < a, \ \vartheta \in [0,2\pi)\}$.

The Laplacian in polar coordinates reads:

$$\Delta w(r,\vartheta) = w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\vartheta\vartheta}.$$

We want to solve the Laplace equation in D_a:

$$\Delta u = 0 \implies \Delta w = 0 \implies w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\vartheta\vartheta} = 0$$

The method of separation of variables starts by writing w as a product of functions that depend only on one variable each: $\underline{w(r,\vartheta) = R(r)\,\Theta(\vartheta)}$.

We use this ansatz to <u>reduce our PDE in a system of ODEs</u> depending by a constant λ.

Starting from the ansatz $w(r,\vartheta) = R(r)\,\Theta(\vartheta)$, we plug this expression in $\Delta w = 0$ and we get:

$$0 = R''(r)\,\Theta(\vartheta) + \frac{1}{r} R'(r)\,\Theta(\vartheta) + \frac{1}{r^2}\,\Theta''(\vartheta) R(r) \implies$$

$$\frac{r^2 R''(r) + r R'(r)}{R(r)} = -\frac{\Theta''(\vartheta)}{\Theta(\vartheta)} = \lambda$$

for which the ODEs system is:

$$(1)\begin{cases} r^2 R''(r) + r R'(r) = \lambda R(r) \\ \Theta''(\vartheta) = -\lambda\,\Theta(\vartheta), \quad \underset{\text{(periodicity)}}{\Theta(0) = \Theta(2\pi)}, \quad \underset{\text{(periodicity of } \Theta')}{\Theta'(0) = \Theta'(2\pi)} \end{cases}$$

Note that since $\Theta'' = -\lambda\,\Theta$, if Θ is 2π-periodic, also Θ'' is periodic. This comes from the fact that we look for classical solutions in D_a (at least C^2).

Solution of (1):

$$(2)\begin{cases} R_n(r) = \begin{cases} C_0 + D_0 \log(r), & n = 0 \\ C_n r^n + D_n r^{-n}, & n \neq 0 \end{cases} \\ \\ \Theta_n(\vartheta) = A_n \cos(n\vartheta) + B_n \sin(n\vartheta), \quad \lambda_n = +n^2. \end{cases}$$

Remark: the functions r^{-n}, $n \in \mathbb{N}$ and $\log(r)$ are singular at the origin $(r=0)$. Therefore, if the origin is in the domain (as in the case of D_a), then we need to discard those solutions imposing $D_n = 0 \;\forall n$.

The general solution is therefore:

$$w(r,\vartheta) = C_0 + \sum_{n=1}^{+\infty} r^n \left[A_n \cos(n\vartheta) + B_n \sin(n\vartheta)\right].$$

We use the boundary conditions to find C_0, A_n, B_n.

Lecture 13

Laplace equation in polar coordinates

Let me remind that a function $u(x,y) \in \mathbb{R}^2$ in polar coordinates takes the form

$$w(r,\vartheta) = u(r\cos\vartheta, r\sin\vartheta), \text{ where}$$

$$\begin{cases} x = r\cos\vartheta \\ y = r\sin\vartheta \end{cases}, \quad r = \sqrt{x^2+y^2}, \quad \vartheta = \arctan\left(\frac{y}{x}\right).$$

The Laplace operator in polar coordinates is:

$$\Delta w(r,\vartheta) = w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\vartheta\vartheta} \qquad (1)$$

We studied the Laplace equation in rectangular domains with some symmetries (for example the boundary data has to be zero on two opposite sides) and we used the method of separation of variables to find the solution. The same method can be applied to domains that are discs, circles, rings or sectors of a circle/ring, but using polar coordinates.

In the case of the Laplacian in polar coordinates, we would look for a solution of the form: $w(r,\vartheta) = R(r)\Theta(\vartheta)$. By plugging this in the formula (1) we would get the relation:

$$r^2 \frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = -\frac{\Theta''(\vartheta)}{\Theta(\vartheta)} = \lambda \in \mathbb{R}$$

For Θ this leads to the simple ODE $\Theta''(\vartheta) = -\lambda\Theta(\vartheta)$.

* If the domain is either a disc or a ring the function Θ must be 2π-periodic$^{(*)}$, as well as Θ', leading to solutions of the form:

$$\Theta_n(\vartheta) = A_n\cos(n\vartheta) + B_n\sin(n\vartheta), \quad \lambda_n = n^2, \ n \in \mathbb{N}.$$

Note that $\Theta_0(\vartheta) = A_0$.

$$\left((*) \rightsquigarrow \begin{cases} \Theta(0) = \Theta(2\pi) \\ \Theta'(0) = \Theta'(2\pi) \end{cases} \right)$$

* In cases where the domain is only a sector of a disc or a ring, then Θ is not necessarily periodic anymore. In this case we will have boundary conditions on Θ.

For R we have the ODE $r^2 R''(r) + r R'(r) = n^2 R(r)$

with solutions:

$$\begin{cases} C_n r^n + D_n r^{-n}, & \text{for } n > 0 \\ C_0 + D_0 \log r, & \text{for } n = 0. \end{cases}$$

In cases where the origin is inside the domain, all D_n are set to zero because we are looking for a smooth solution in the domain.

Example: Let $u = u(x,y)$ and $B_1 = \{x^2 + y^2 \leq 1\}$ be the unit disc in $\mathbb{R}^2$. We want to solve the following Dirichlet problem:

$$\begin{cases} \Delta u = 0 & \text{in } B_1 \\ u = y^2 & \text{on } \partial B_1. \end{cases}$$

Using polar coordinates $w(r,\theta) = u(r\cos\theta, r\sin\theta)$ we can rewrite the problem as:

$$\begin{cases} \Delta w = w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\theta\theta} = 0, & r \in (0,1), \theta \in (0, 2\pi). \\ w(1,\theta) = \sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2\theta). \end{cases}$$

The solution we look for has the form:

$w(r,\theta) = R(r)\,\Theta(\theta)$, and as seen before the solution for R is: $R_n(r) = C_n r^n$ if $n > 0$ and $R_0(r) = C_0$.

For Θ we take the general solution :

$$\Theta_n(\vartheta) = A_n \cos(n\vartheta) + B_n \sin(n\vartheta) , \quad \Theta_0(\vartheta) = A_0 .$$

Thus, up to renaming the coefficients, we find the general solution :

$$w(r,\vartheta) = E + \sum_{n=1}^{+\infty} \left[A_n \cos(n\vartheta) + B_n \sin(n\vartheta) \right] r^n .$$

Now let's use the boundary condition:

$$\frac{1}{2} - \frac{1}{2}\cos(2\vartheta) = w(1,\vartheta) = E + \sum_{n=1}^{+\infty} \left[A_n \cos(n\vartheta) + B_n \sin(n\vartheta) \right].$$

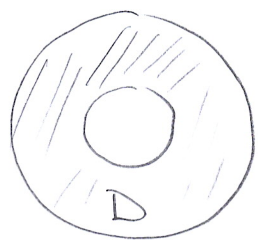
From this equality we deduce that $E = \frac{1}{2}$, $A_2 = -\frac{1}{2}$ and all other coefficients A_n, B_n are zero. Thus, the final solution is :

$$w(r,\vartheta) = \frac{1}{2} - \frac{1}{2} r^2 \cos(2\vartheta).$$

(Exercise : check that if you come back in Cartesian coordinates the solution is $u(x,y) = \frac{1}{2}(1 - x^2 + y^2)$.)

Example : let us now consider the following problem:

$$\begin{cases} \Delta u = 0 \quad \text{on} \quad D = \{ (x,y) \in \mathbb{R}^2 : 1 \le \sqrt{x^2+y^2} \le 2 \} \\ + \text{ Dirichlet boundary conditions} \quad \text{on} \quad \partial D \end{cases}$$



In polar coordinates the problem reads as :

$$\begin{cases} \Delta w = 0 \quad \text{in} \quad D = \{ r \in (1,2), \vartheta \in [0,2\pi) \} \\ w(2,\vartheta) = 3\cos\vartheta, \quad \text{for } r=2 \\ w(1,\vartheta) = \sin\vartheta, \quad \text{for } r=1 \end{cases}$$

The ODEs for R and Θ are the same as before, 13.4 but now the boundary conditions for R have changed.

$$r^2 R''(r) + r R'(r) = n^2 R(r), \quad \lambda_n = n^2, \quad n \in \mathbb{N}.$$

The solution is given by:

$$\begin{cases} C_n r^n + D_n r^{-n}, & n>0 \\ C_0 + D_0 \log r & \text{for } n=0. \end{cases}$$

Since now the origin is not contained in D, we cannot discard the (otherwise singular) solutions r^{-n} and $\log(r)$. Therefore $w(r,\theta) = \sum_n R_n(r)\Theta_n(\theta)$ takes the form:

$$w(r,\theta) = E + F\log r + \sum_{n=1}^{+\infty}\Big[A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta) + C_n r^{-n}\cos(n\theta) + $$
$$+ D_n r^{-n}\sin(n\theta)\Big].$$

Using the boundary condition $w(1,\theta) = \sin\theta$, we have:

$$\sin\theta = w(1,\theta) = E + \sum_{n=1}^{+\infty}\Big[(A_n + C_n)\cos(n\theta) + (B_n + D_n)\sin(n\theta)\Big]$$

This implies: $E = 0$, $B_1 + D_1 = 1$, $B_n + D_n = 0 \quad \forall\, n \geq 2$,
$A_n + C_n = 0 \quad \forall\, n \geq 1$.

By the condition $w(2,\theta) = 3\cos(\theta)$ we have:

$$3\cos\theta = w(2,\theta) = E + F\log(2) + \sum_{n=1}^{\infty}\Big[(2^n A_n + 2^{-n}C_n)\cos n\theta +$$
$$(2^n B_n + 2^{-n}D_n)\sin(n\theta)\Big].$$ This implies: $E + F\log(2) = 0$

$2^n B_n + 2^{-n} D_n = 0 \quad \forall n \geq 1$, $2A_1 + \frac{1}{2}C_1 = 3$, $2^n A_n + 2^{-n} C_n = 0$ for $n \geq 2$.

Combining all these information:

$E=0$, $E+F\log(2)=0 \Rightarrow E=F=0$.

$A_1+C_1=0$, $2A_1+\frac{1}{2}C_1=3 \Rightarrow A_1=2$, $C_1=-2$

$B_1+D_1=1$, $2B_1+\frac{1}{2}D_1=0 \Rightarrow B_1=-\frac{1}{3}$, $D_1=\frac{4}{3}$

Also, for $n \geq 2$:

$A_n+B_n=0$, $2^n A_n+2^{-n}C_n=0 \Rightarrow A_n=C_n=0$.

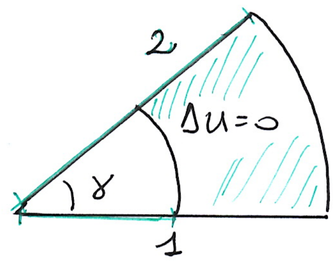
$B_n+D_n=0$, $2^n B_n+2^{-n}D_n=0 \Rightarrow B_n=D_n=0$.

This proves that:

$$w(r,\vartheta)=2r\cos\vartheta-\frac{1}{3}r\sin\vartheta-2r^{-1}\cos\vartheta+\frac{4}{3}r^{-1}\sin\vartheta.$$

Example: Let us now solve the Laplace equation on an anular sector of angle $\gamma \in (0,2\pi)$ and radii 1 and 2, i.e:

$D=\{(r,\vartheta): r\in(1,2),\ \vartheta\in(0,\gamma)\}$.



An anular sector in $\mathbb{R}^2$.

To solve such problem we rely on the formula for the Laplacian in polar coordinates and on the method of separation of variables. We look for a solution of the form $w(r,\vartheta)=R(r)\Theta(\vartheta)$. The PDE gives

$$\begin{cases} r^2R''(r)+rR'(r)=+\lambda R(r), \\ \Theta''(\vartheta)=-\lambda\,\Theta(\vartheta) \end{cases}$$

Assume that w is prescribed on ∂D (Dirichlet problem) and assume that

$w(r,0)=0$, $w(r,\gamma)=0$ for all $r\in(1,2)$.

Then, to enforce these boundary conditions we impose $\Theta(0)=\Theta(\gamma)=0$. Hence we have

$$\begin{cases} \Theta''(\vartheta) = -\lambda\, \Theta(\vartheta) \\ \Theta(0) = \Theta(\gamma) = 0 \end{cases} \Rightarrow \Theta_n(\vartheta) = A_n \sin\left(\frac{n\pi}{\gamma}\vartheta\right), \quad \lambda_n = \left(\frac{n\pi}{\gamma}\right)^2.$$

Then the ODE for R_n becomes

$$r^2 R_n''(r) + r R_n'(r) - \lambda_n R_n(r) = 0.$$

Looking for solutions of the form r^α we get:

$$0 = \alpha(\alpha-1) + \alpha - \lambda_n = \alpha^2 - \lambda_n \Rightarrow \alpha = \pm\sqrt{\lambda_n}.$$

Hence $R_n(r) = C_n r^{\frac{n\pi}{\gamma}} + D_n r^{-\frac{n\pi}{\gamma}}$ and the general solution in this case is given by:

$$w(r,\vartheta) = \sum_{n\geq 1} A_n \sin\left(\frac{n\pi}{\gamma}\vartheta\right) r^{\frac{n\pi}{\gamma}} + B_n \sin\left(\frac{n\pi}{\gamma}\vartheta\right) r^{-\frac{n\pi}{\gamma}},$$

and the coefficients A_n and B_n are found expanding the boundary conditions $w(1,\vartheta)$ and $w(2,\vartheta)$ over the interval $\vartheta \in [0,\gamma]$ using the Fourier basis $\left\{\sin\left(\frac{n\pi}{\gamma}\vartheta\right)\right\}$.

Remark: If the sector is $D = \{(r,\vartheta),\ r \in [0,2),\ \vartheta \in (0,\gamma)\}$ with boundary conditions $w(r,0) = 0$ and $w(r,\gamma) = 0$ for all $r \in (0,2)$, then the general solution is of the form:

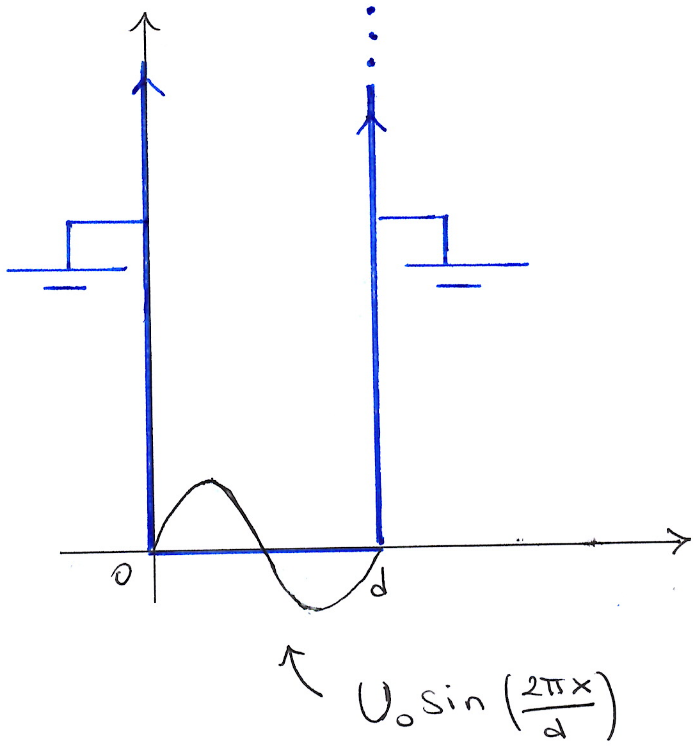
$$w(r,\vartheta) = \sum_{n\geq 1} A_n \sin\left(\frac{n\pi}{\gamma}\vartheta\right) r^{\frac{n\pi}{\gamma}}$$

since the negative powers of r are singular at the origin and should be discarded.

A "real life" example (non examinable).

Consider a pair of infinite, grounded conducting sheets separated at distance d.
Suppose that there is a conductor connecting the two metal sheets held at $U_0 \sin\left(\frac{2\pi x}{d}\right)$ as in figure.
What is the potential between the plates?

We know that the electric potential satisfies the Laplace equation in the region between plates (no charge in there).
Therefore we want to solve the following Dirichlet problem:



$$\begin{cases} \Delta u = 0 \text{ in } (0,d)\times \mathbb{R}^+ \\ u(x,0) = U_0 \sin\left(\frac{2\pi x}{d}\right) \\ u(0,y) = u(d,y) = 0 \end{cases}$$

Note that there in an additional implicit boundary condition : we would like the potential to go to zero in the "open" spatial direction, that translates into

$$\lim_{y\to+\infty} u(x,y) = 0 .$$

Let suppose that $u(x,y) = \sum_n X_n(x)\, Y_n(y)$.

$$0 = \Delta u(x,y) = \sum_n X_n''(x)\, Y_n(y) + X_n(x)\, Y_n''(y) .$$

term by term:

$$\frac{X_n''(x)}{X_n(x)} = -\frac{Y_n''(y)}{Y_n(y)} = -\lambda_n \in \mathbb{R}.$$

This leads us to the two ODEs:

$$\begin{cases} X_n''(x) = -\lambda_n X_n(x), \quad X_n(0) = X_n(d) = 0 \\ Y_n''(y) = \lambda_n Y_n(y). \end{cases}$$

The solution for the first one is the following:

$X_n(x) = A_n \operatorname{sen}(\sqrt{\lambda_n}\, x)$ with $\lambda_n = \left(\frac{n\pi}{d}\right)^2 \quad \forall n \in \mathbb{N}$.

(we have Dirichlet conditions, thus we can expand in terms of $\operatorname{sen}(\sqrt{\lambda_n}\, x)$ only).

$Y_n(y) = C_n \sinh(\sqrt{\lambda_n}\, y) + D_n \cosh(\sqrt{\lambda_n}\, y)$.

By the condition:

$\lim_{y \to +\infty} Y_n(y) = 0$ we deduce:

$$\lim_{y \to +\infty} \left(C_n \frac{e^{\sqrt{\lambda_n} y} - e^{-\sqrt{\lambda_n} y}}{2} + D_n \frac{e^{\sqrt{\lambda_n} y} + e^{-\sqrt{\lambda_n} y}}{2} \right) =$$

$$\lim_{y \to +\infty} \left(\frac{C_n + D_n}{2}\right) e^{\sqrt{\lambda_n}\, y} + \left(\frac{D_n - C_n}{2}\right) e^{-\sqrt{\lambda_n}\, y} = 0$$

Therefore we have that $C_n + D_n = 0 \quad \forall n \Rightarrow D_n = -C_n$.

Thus: $Y_n(y) = D_n\left(\cosh(\sqrt{\lambda_n}\, y) - \sinh(\sqrt{\lambda_n}\, y)\right) = D_n e^{-\sqrt{\lambda_n}\, y}$.

The general solution is given by:

$u(x,y) = \sum_n A_n \operatorname{sen}\left(\frac{n\pi x}{d}\right) e^{-\frac{n\pi}{d} y}$. By the condition $u(x,0) = U_0 \sin\left(\frac{2\pi x}{d}\right)$ we deduce that $A_n = 0 \quad \forall\, n \neq 2$ and the final solution is

$$\underline{u(x,y) = U_0 \sin\left(\frac{2\pi x}{d}\right) e^{-\frac{2\pi y}{d}}}$$