BASIC PROPERTIES OF THE FOURIER TRANSFORM

The Fourier transform of an integrable function $f : \mathbb{R} \to \mathbb{R}$ is defined as

$$\mathcal{F}(f)(\xi) := \int_{\mathbb{R}} f(x) e^{-ix\xi}\, d\xi.$$

Theorem. *The Fourier transform satisfies the following basic properties:*

Translation: *Given $a \in \mathbb{R}$, let $\tau_a f(t) := f(t-a)$. If f is integrable, we have*

$$\mathcal{F}[\tau_a f](\xi) = \mathrm{e}^{-i\xi a}\mathcal{F}[f](\xi),$$

$$\tau_a \mathcal{F}[f](\xi) = \mathcal{F}[\mathrm{e}^{iax} f(x)](\xi).$$

Dilation: *Given $\lambda > 0$, $\delta_\lambda f(x) := f(\lambda x)$. If f is integrable, we have*

$$\mathcal{F}[\delta_\lambda f](\xi) = \frac{1}{\lambda}\delta_{1/\lambda}\mathcal{F}[f](\xi),$$

$$\delta_\lambda \mathcal{F}[f](\xi) = \frac{1}{\lambda}\mathcal{F}\left[\delta_{1/\lambda} f(x)\right](\xi).$$

Derivative: *If f, f' and $x\, f(x)$ are integrable, then we have*

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{F}[f](\xi) = (-i)\mathcal{F}\left[x f(x)\right](\xi),$$

$$\mathcal{F}\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right](\xi) = i\xi\mathcal{F}[f](\xi).$$

Higher order derivative: *Let $k \geq 1$ be a positive integer. If f, f', f'', $\ldots$, $f^{(k)}$ and $xf(x)$, $x^2 f(x)$, $\ldots$, $x^k f(x)$ are integrable, then we have*

$$\frac{\mathrm{d}^k}{\mathrm{d}\xi^k}\mathcal{F}[f](\xi) = (-i)^k\mathcal{F}\left[x^k f(x)\right](\xi),$$

$$\mathcal{F}\left[\frac{\mathrm{d}^k f}{\mathrm{d}x^k}\right](\xi) = (i\xi)^k\mathcal{F}[f](\xi).$$

Convolution: *If f and g are integrable functions, it holds $\mathcal{F}[f * g] = \mathcal{F}[f]\mathcal{F}[g]$.*

Proof. We prove the various properties independently one from the other.

Translation: Since $\int_{\mathbb{R}} |\tau_a f| \, \mathrm{d}x = \int_{\mathbb{R}} |f| \, \mathrm{d}x$, $\tau_a f$ is integrable, and by change of variable $y = x - a$ we have

$$\begin{aligned}\mathcal{F}[\tau_a f](\xi) &= \int_{\mathbb{R}} f(x-a)\mathrm{e}^{-i\xi x} \, \mathrm{d}x \\ &= \int_{\mathbb{R}} f(y)\mathrm{e}^{iay}e^{-i\xi y} \, \mathrm{d}y = \mathcal{F}[f(x)\mathrm{e}^{iax}](\xi).\end{aligned}$$

Then

$$\begin{aligned}\tau_a \mathcal{F}[f](\xi) &= \mathcal{F}[f](\xi - a) \\ &= \int_{\mathbb{R}} f(x)\mathrm{e}^{-i(\xi - x)x} \, \mathrm{d}x = \mathcal{F}[f(x)\mathrm{e}^{iax}](\xi).\end{aligned}$$

Dilation: Because f is integrable, $\delta_\lambda f$ and $\delta_{1/\lambda} f$ are also integrable. By change of variable $y = \lambda x$ we have

$$\begin{aligned}\mathcal{F}[\delta_\lambda f](\xi) &= \int_{\mathbb{R}} f(\lambda x)\mathrm{e}^{-i\xi x} \, \mathrm{d}x \\ &= \int_{\mathbb{R}} f(y)\mathrm{e}^{-i\xi \frac{y}{\lambda}} \frac{dx}{\lambda} = \frac{1}{\lambda}\mathcal{F}[f]\left(\frac{\xi}{\lambda}\right) = \frac{1}{\lambda}\delta_{1/\lambda}\mathcal{F}[f](\xi),\end{aligned}$$

and

$$\begin{aligned}\delta_\lambda \mathcal{F}[f](\xi) &= \mathcal{F}[f](\lambda \xi) \\ &= \int_{\mathbb{R}} f(x)\mathrm{e}^{-i\lambda\xi x} \, \mathrm{d}x = \int_{\mathbb{R}} f\left(\frac{y}{\lambda}\right)\mathrm{e}^{-i\xi y}\frac{\mathrm{d}y}{\lambda} = \frac{1}{\lambda}\mathcal{F}[\delta_{1/\lambda} f](\xi).\end{aligned}$$

(Higher order) Derivative: We compute:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{F}[f](\xi) &= \frac{\mathrm{d}}{\mathrm{d}\xi}\int_{\mathbb{R}} f(x)\mathrm{e}^{-i\xi x} \, \mathrm{d}x = \int_{\mathbb{R}} f(x)(-ix)\mathrm{e}^{-i\xi x} \, \mathrm{d}x \\ &= -i\mathcal{F}[xf(x)](\xi).\end{aligned}$$

By partial integration we have:

$$\begin{aligned}\mathcal{F}[f'](\xi) &= \int_{\mathbb{R}} f'(x)\mathrm{e}^{-i\xi x} \, \mathrm{d}x \\ &= f(x)\mathrm{e}^{-i\xi x}\Big|_{-\infty}^{+\infty} - \int_{\mathbb{R}} f(x)\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{-i\xi x}) \, \mathrm{d}x \\ &= \lim_{k\to\infty} f(R_k)\mathrm{e}^{-i\xi R} - \lim_{k\to\infty} f(-R_k)\mathrm{e}^{-i\xi R} + i\xi\mathcal{F}[f](\xi).\end{aligned}$$

Since f is integrable, we have a sequence of positive real numbers $\{R_k\}_{k\in\mathbb{N}}$ such that $\lim_{k\to+\infty} R_k = \infty$, $\lim_{k\to\infty} f(R_k) = 0$ and $\lim_{k\to\infty} f(-R_k) = 0$. Note that $|\mathrm{e}^{-i\xi R}| = 1$, we can deduce $\lim_{R\to\pm\infty} f(R)\mathrm{e}^{-i\xi R} = 0$. Therefore,

$$\mathcal{F}[f'](\xi) = i\xi\mathcal{F}[f](\xi).$$

For the higher order derivatives, we have

$$\begin{aligned}\frac{\mathrm{d}^k}{\mathrm{d}\xi^k}\mathcal{F}[f](\xi) &= \frac{\mathrm{d}^k}{\mathrm{d}\xi^k}\int_{\mathbb{R}} f(x)\mathrm{e}^{-i\xi x}\,\mathrm{d}x = \int_{\mathbb{R}} f(x)(-ix)^k\mathrm{e}^{-i\xi x}\,\mathrm{d}x \\ &= (-i)^k\mathcal{F}[x^k f(x)](\xi).\end{aligned}$$

For the last step we use partial integration k times in the following way:

$$\begin{aligned}\mathcal{F}[f^{(k)}](\xi) &= \int_{\mathbb{R}} f^{(k)}(x)\mathrm{e}^{-i\xi x}\,\mathrm{d}x = \underbrace{\left. f^{(k-1)}(x)\mathrm{e}^{-i\xi x}\right|_{-\infty}^{+\infty}}_{=0,\text{ since } f^{(k-1)} \text{ is integrable}} +(i\xi)\int_{\mathbb{R}} f^{(k-1)}\mathrm{e}^{-i\xi x}\,\mathrm{d}x \\ &= \underbrace{\left.(i\xi)f^{(k-2)}\mathrm{e}^{-i\xi x}\right|_{-\infty}^{+\infty}}_{=0,\text{ since } f^{(k-2)} \text{ is integrable}} +(i\xi)^2\int_{\mathbb{R}} f^{(k-2)}\mathrm{e}^{-i\xi x}\,\mathrm{d}x \\ &= \dots \\ &= (i\xi)^k\int f(x)\mathrm{e}^{-i\xi x}\,\mathrm{d}x = (i\xi)^k\mathcal{F}[f](\xi).\end{aligned}$$

Convolution: Because f and g are integrable, $f * g$ is integrable. For $\xi \in \mathbb{R}$:

$$\begin{aligned}\mathcal{F}[f * g](\xi) &= \int_{\mathbb{R}}(f * g)(x)\mathrm{e}^{-i\xi x}\,\mathrm{d}x = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-y)g(y)\,\mathrm{d}y\ \mathrm{e}^{-i\xi x}\,\mathrm{d}x \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-y)\mathrm{e}^{-i\xi(x-y)}\,\mathrm{d}x\ g(y)\mathrm{e}^{-i\xi y}\,\mathrm{d}y \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(z)\mathrm{e}^{-i\xi z}\,\mathrm{d}z\ g(y)\mathrm{e}^{-i\xi y}\,\mathrm{d}y = \mathcal{F}[f](\xi)\mathcal{F}[g](\xi).\end{aligned}$$

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