# ANALYSIS III: PDE - LAST EXERCISE CLASS

**Problem 1.** Let $D := \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 < 9\}$. Consider the problem

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u(x, y) = 7 + 5xy + 2y^2 & \text{on } \partial D. \end{cases}$$

(1) Determine $u(0, 0)$;
(2) Write down the solution both in polar coordinates $(r, \theta)$ and in Cartesian coordinates $(x, y)$.

*Solution.* Step 1: Calculate $u(0, 0)$

In order to calculate $u(0, 0)$ we are going to use the Mean Value Property of harmonic functions. Remembering the following trigonometric identities:

$$\cos\theta \sin\theta = \frac{1}{2}\sin\theta$$
$$(\sin\theta)^2 = \frac{1}{2}(1 - \cos 2\theta)$$

one has that the boundary datum $g(x, y) = 7 + 5xy + 2y^2$ can be written in polar coordinates (for $r = 3$) as

$$\begin{aligned} f(\theta) =& 7 + \frac{45}{2}\sin 2\theta + 9(1 - \cos 2\theta) \\ =& 16 + \frac{45}{2}\sin 2\theta - 9\cos 2\theta \end{aligned}$$

As a result, we get

$$u(0, 0) = 16 + \frac{1}{2\pi}\int_0^{2\pi} \frac{45}{2}\sin 2\theta \, d\theta + \frac{1}{2\pi}\int_0^{2\pi} (-9)\cos 2\theta \, d\theta = 0$$

since (by periodicity) clearly

$$\int_0^{2\pi} \frac{45}{2}\sin 2\theta \, d\theta = 0, \text{ and } + \int_0^{2\pi} (-9)\cos 2\theta \, d\theta = 0.$$

Thus we conclude that

$$u(0, 0) = 16 \tag{1}$$

This finishes part (1).

Step 2: Solving for $u$ in both cartesian and polar coordinates

Recall from the lecture that the general solution to $\Delta u = 0$ on $B$ is given in polar coordinates i.e. for

$$\begin{cases} x = & r\cos\theta \\ y = & r\sin\theta \end{cases}$$

we get

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left(a_n \cos n\theta + b_n \sin n\theta\right)$$

As a next step we want to solve by using the boundary conditions

$$\begin{aligned} u(3,\theta) =& 7 + 5xy + 2y^2 \\ =& 7 + 5(3\cos\theta)(3\sin\theta) + 2(3\sin\theta)^2 =: f(\theta) \end{aligned}$$

We want to choose $a_n, b_n$ such that the following equality holds:

$$u(3,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} 3^n \left(a_n \cos n\theta + b_n \sin n\theta\right) \overset{!}{=} 16 + \frac{45}{2}\sin 2\theta - 9\cos 2\theta$$

By uniqueness of Fourier series we get $a_0 = 32, a_2 = -1, b_2 = \frac{5}{2}$, while all other $a_n$ and $b_n$ are zero. Thus our final solution is

$$u(r,\theta) = 16 + r^2 \left(-\cos 2\theta + \frac{5}{2}\sin 2\theta\right)$$

As a last step we need to transform this back into Cartesian coordinates. Again remember the following trigonometric identities:

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ \sin 2x &= 2\sin x \cos x \end{aligned}$$

Using both identities we conclude

$$\begin{aligned} u(x,y) =& 16 + r^2 \left(-\cos 2\theta + \frac{5}{2}\sin 2\theta\right) \\ =& 16 + r^2 \left(-(\cos^2\theta - \sin^2\theta) + 5\sin\theta\cos\theta\right) && \text{(Using both identities)} \\ =& 16 + r^2 \left(\sin^2\theta - \cos^2\theta + 5\sin\theta\cos\theta\right) \\ =& 16 + y^2 - x^2 + 5xy && \text{(Using Polar Coordinates)} \end{aligned}$$

This finishes the problem. □

**Problem 2.** Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = xe^{-4(x-1)^2}$.

(1) Compute the Fourier transform of $f$.
(2) Using the previous part of this exercise, compute the value of $\int_{-\infty}^{+\infty}(1+x)f(x)\,dx$.

*Solution.* Recall from Lecture 6 that for $g(x) = e^{-\frac{x^2}{2}}$ we have $\mathcal{F}[g](\xi) = \sqrt{2\pi}e^{-\frac{\xi^2}{2}}$.

Step 1: Calculating the Fourier Transform

$$\begin{aligned}\mathcal{F}[f](\xi) &= \int_{\mathbb{R}} xe^{-4(x-1)^2}e^{-ix\xi}dx \\ &= \int_{\mathbb{R}} (x+1)e^{-4x^2}\underbrace{e^{-i(x+1)\xi}}_{e^{-i\xi}e^{-ix\xi}}dx \qquad \text{(Using } x = u+1 \text{ and } dx = du) \\ &= e^{-i\xi}\left(\underbrace{\int_{\mathbb{R}} xe^{-4x^2}e^{-ix\xi}dx}_{I_1} + \underbrace{\int_{\mathbb{R}} e^{-4x^2}e^{-ix\xi}dx}_{I_2}\right)\end{aligned}$$

For the first integral we are going to make use of Property $B$ and Property $C$ from Lecture 6. First of all, for $h(x) = e^{-4x^2}$ we get

$$I_1 = \int_{\mathbb{R}} xe^{-4x^2}e^{-ix\xi}dx = \int_{\mathbb{R}} xh(x)e^{-ix\xi}dx = \mathcal{F}[xh](\xi) \stackrel{B}{=} i\frac{d}{d\xi}\mathcal{F}[h](\xi)$$

Using the fact that $h(x) = g(\sqrt{8}x)$ we further know that

$$\mathcal{F}[h](\xi) \stackrel{C}{=} \frac{1}{\sqrt{8}}\mathcal{F}[g]\left(\frac{\xi}{\sqrt{8}}\right) = \frac{1}{\sqrt{8}}\sqrt{2\pi}e^{-\frac{\xi^2}{2\sqrt{8}^2}} = \frac{\sqrt{\pi}\sqrt{2}}{2\sqrt{2}}e^{-\frac{\xi^2}{16}} = \frac{\sqrt{\pi}}{2}e^{-\frac{\xi^2}{16}}$$

Using this we finish the computation of $I_1$:

$$\begin{aligned}I_1 &= i\frac{d}{d\xi}\mathcal{F}[h](\xi) = i\frac{d}{d\xi}\left(\frac{\sqrt{\pi}}{2}e^{-\frac{\xi^2}{16}}\right) \\ &= -i\frac{\sqrt{\pi}}{2}\frac{2\xi}{16}e^{-\frac{\xi^2}{16}} = \frac{-i\xi\sqrt{\pi}}{16}e^{-\frac{\xi^2}{16}}\end{aligned}$$

For $I_2$ we just reuse the same trick again, thus

$$I_2 = \mathcal{F}[h](\xi) = \mathcal{F}[g]\left(\frac{\xi}{\sqrt{8}}\right) = \frac{\sqrt{\pi}}{2}e^{-\frac{\xi^2}{16}}$$

Now combining $I_1$ and $I_2$ yields

$$I_1 + I_2 = \sqrt{\pi}\left(\frac{8-i\xi}{16}\right)e^{-\frac{\xi^2}{16}}$$

And so we finally get

$$\mathcal{F}[f](\xi) = e^{-i\xi}\left(\sqrt{\pi}\left(\frac{8-i\xi}{16}\right)e^{-\frac{\xi^2}{16}}\right)$$

Step 2: Use this in order to calculate $\int_{\mathbb{R}}(x+1)f(x)dx$

Again we make use of the properties of the Fourier transformed presented in lecture 6.

$$\int_{\mathbb{R}}(x+1)f(x)e^{-ix0}dx = \mathcal{F}[(x+1)f(x)](0) \overset{lin}{=} \underbrace{\mathcal{F}[xf(x)](0)}_{I_1} + \underbrace{\mathcal{F}[f(x)](0)}_{I_2}$$

For $I_1$ we again use Property $B$

$$\mathcal{F}[xf(x)](\xi) \overset{B}{=} (i\frac{d}{d\xi}\mathcal{F}[f](\xi)) \overset{Calc}{=} \frac{\sqrt{\pi}}{16}\left(9 - 2i\xi - \frac{\xi^2}{8}\right)e^{-\xi^2/16 - i\xi}.$$

For $I_2$ we directly get by (1) that $I_2 = (\sqrt{\pi}(\frac{8}{16})) = \frac{1}{2}\sqrt{\pi}$. Thus finally we get

$$\int_{\mathbb{R}}(x+1)f(x)dx = I_1 + I_2 = \sqrt{\pi}\frac{17}{16}$$

This finishes the problem. □