

Mathematik III

Probeproofung

Identifying number: 001

- Put your student identity card on the desk. **Do not open this booklet** until you are so instructed.
- The exam is pseudonymised: this means that you do **not** have to write your name or your student number on the exam, but you are identified by the number we told you prior to the exam. Please check the correctness of the identifying number above before you start working!
- During the exam no written aids nor calculators or any other electronic device are allowed. **Phones must be switched off and stowed away** in your bag during the whole duration of the exam.
- A4-paper is provided. No other paper is allowed. Write with blue or black pens. **Do not use pencils, erasable pens, red or green ink, nor Tipp-Ex.**
- Write your solution to each exercise in the designated white space in the booklet. Leave enough (≈ 2 cm) empty space on the margins (top, bottom and sides). If you need more space to write your solutions you can use the provided A4-paper, but please start every part on a new sheet of paper and **write your identifying number on every sheet of paper.**
- Please write neatly! Please do not put the graders in the unpleasant situation of being incapable of reading your solutions, as this will certainly not play in your favour!
- All your answers, in any section of the exam, do need to be properly justified. It is fine and allowed to use theorems/statements proved in class or in the homework (i. e. in Problem Sets 1–12) without reproving them (**unless otherwise stated**), but you should provide a precise statement of the result in question.
- Said $0 \leq s \leq 60$ your total score on the exercises, your final grade in the exam will be *bounded from below* by $\frac{s}{10}$.
- The duration of the exam is **120 minutes**.

Do not fill out this table!

part	points	check
1	[10]	
2	[10]	
3	[20]	
4	[20]	
total	[60]	

grade:

Part 1

[10 points]

For each of the following 5 statement you have to establish whether it is true or false. Each correct answer is worth 2 points. An empty answer is worth 0 points. A wrong answer detracts 2 points (that is, it is worth -2 points). If your total score in this exercise is negative, you will be given 0 (so, it is impossible to get a negative score on the whole exercise).

Insert your answers in the following grid. Write clearly **T** if the statement is true and **F** if the statement is false. We will accept also **R** if the statement is *richtig* (which is the German word for *true*).

Only the answers in the grid will be taken into consideration for grading.

Question	1	2	3	4	5
Answer					

Consider the exponential Fourier series of a real-valued continuous 2π -periodic function $f : \mathbb{R} \rightarrow \mathbb{R}$:

$$f(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx}.$$

1. If $c_0 = 0$, then

$$\int_{-\pi}^{\pi} f(x) dx = 0.$$

2. If c_n is purely imaginary for all $n \in \mathbb{Z}$, then f is an even function.

3. If $c_{-1} = c_1 = 0$ then

$$f\left(-\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right)$$

4. For any fixed complex number $z_0 \in \mathbb{C}$, there exists at least one continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\sum_{n \in \mathbb{N}} c_n = z_0.$$

5. If $c_n = 0$ for all $n \in \mathbb{Z}$ such that $|n| > 100$, then f is differentiable.

Part 2

[10 points]

For each of the following 5 questions, you have to provide a *numeric or symbolic answer*. An example of a numeric answer is $-27\sqrt{2}$; an example of a symbolic answer is a^3/b . Each correct answer is worth 2 points. An empty or wrong answer is worth 0 points.

Insert your answers in the following grid. Write clearly so that there is no ambiguity. Only the answers in the grid will be taken into consideration for grading. You do **not** have to provide any justification for your answers.

Question	1	2	3	4	5
Answer					

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an integrable function such that its Fourier transform equals

$$\hat{f}(\xi) = \frac{e^{-(a+i\xi)}}{a+i\xi}$$

where $a > 0$ is a positive parameter, understood to be unspecified throughout the exercise.

1. As a function of $a > 0$, what is the value of $\int_{-\infty}^{+\infty} f(x) dx$?
2. As a function of $a > 0$, what is the value of $\int_{-\infty}^{+\infty} xf(x) dx$?
3. As a function of $a > 0$, what is the Fourier transform of the function $x \mapsto f(x+5)$?
4. In the case when $a = 1$ what is the Fourier transform of the convolution $f * f$?
5. In the case when $a = -3$ what is the value of $\mathcal{F}(f')(1)$?

Part 3

[20 points]

Let $u : [0, 2] \times [0, \infty) \rightarrow \mathbb{R}$ be the solution of the following PDE:

$$\begin{cases} u_t - 5u_{xx} = 0 & \text{for } x \in (0, 2), t > 0, \\ u(0, t) = 3 & \text{for } t > 0, \\ u(2, t) = 7 & \text{for } t > 0, \\ u(x, 0) = 2\left(x + \frac{3}{2}\right) - \frac{\sin(\pi x)}{2} - 4 \sin(\pi x) \cos(\pi x) & \text{for } x \in (0, 2). \end{cases}$$

- (i) Find a function $v(x, t) = v(x)$ which depends only on the x variable that solves

$$\begin{cases} v_t - 5v_{xx} = 0 & \text{for } x \in (0, 2), t > 0, \\ v(0, t) = 3 & \text{for } t > 0, \\ v(2, t) = 7 & \text{for } t > 0. \end{cases}$$

- (ii) Write down the PDE (and its initial and boundary conditions) solved by $w := v - u$ (where u is the original PDE and v is the function constructed in the previous step).
- (iii) Compute the Fourier series *in sines only* of the function $\frac{\sin(\pi x)}{2} + 4 \sin(\pi x) \cos(\pi x)$.
- (iv) With the separation of variables method, find an explicit formula for the function w and deduce from it an explicit expression for the function u .

Part 4

[20 points]

Let $u : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$ be the solution of the following PDE:

$$\begin{cases} u_{tt}(x, t) = u_{xx}(x, t) & \text{for } x \in \mathbb{R}, t > 0, \\ u(x, 0) = \begin{cases} \frac{1}{1+x} & \text{for } x \geq 0 \\ \frac{1}{1-x} & \text{otherwise} \end{cases}, \\ u_t(x, 0) = \begin{cases} x^2 & \text{for } x \in [-1, 2] \\ 0 & \text{otherwise} \end{cases}. \end{cases}$$

- (i) Compute $u(3, 2)$.
- (ii) For any fixed $x \in \mathbb{R}$, what is the value of $\lim_{t \rightarrow \infty} u(x, t)$?
- (iii) Draw a qualitative graph for the function $x \mapsto u(x, 4)$.
- (iv) Is the function u continuous?

