Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich
Prof. Dr. Alessandro Carlotto 17 November 2021

Mathematik III
Probeprüfung

Identifying number: 001

- Put your student identity card on the desk. **Do not open this booklet** until you are so instructed.
- The exam is pseudonymised: this means that you do **not** have to write your name or your student number on the exam, but you are identified by the number we told you prior to the exam. Please check the correctness of the identifying number above before you start working!
- During the exam no written aids nor calculators or any other electronic device are allowed. **Phones must be switched off and stowed away** in your bag during the whole duration of the exam.
- A4-paper is provided. No other paper is allowed. Write with blue or black pens. **Do *not* use pencils, erasable pens, red or green ink, nor Tipp-Ex.**
- Write your solution to each exercise in the designated white space in the booklet. Leave enough ($\approx 2\,\text{cm}$) empty space on the margins (top, bottom and sides). If you need more space to write your solutions you can use the provided A4-paper, but please start every part on a new sheet of paper and **write your identifying number on every sheet of paper**.
- Please write neatly! Please do not put the graders in the unpleasant situation of being incapable of reading your solutions, as this will certainly not play in your favour!
- All your answers, in any section of the exam, do need to be properly justified. It is fine and allowed to use theorems/statements proved in class or in the homework (i. e. in Problem Sets 1–12) without reproving them (**unless otherwise stated**), but you should provide a precise statement of the result in question.
- Said $0 \leq s \leq 60$ your total score on the exercises, your final grade in the exam will be *bounded from below* by $\frac{s}{10}$.
- The duration of the exam is **120 minutes**.

Do not fill out this table!

part	points	check
1	[10]	
2	[10]	
3	[20]	
4	[20]	
total	[60]	

grade:

Part 1 [10 points]

For each of the following 5 statement you have to establish whether it is true or false. Each correct answer is worth 2 points. An empty answer is worth 0 points. A wrong answer detracts 2 points (that is, it is worth −2 points). If your total score in this exercise is negative, you will be given 0 (so, it is impossible to get a negative score on the whole exercise).

Insert your answers in the following grid. Write clearly **T** if the statement is true and **F** if the statement is false. We will accept also **R** if the statement is *richtig* (which is the German word for *true*).

Only the answers in the grid will be taken into consideration for grading.

Question	1	2	3	4	5
Answer					

Consider the exponential Fourier series of a real-valued continuous 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$:

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx}.$$

1. If $c_0 = 0$, then

$$\int_{-\pi}^{\pi} f(x)\, dx = 0.$$

2. If c_n is purely imaginary for all $n \in \mathbb{Z}$, then f is an even function.

3. If $c_{-1} = c_1 = 0$ then

$$f\Big(-\frac{\pi}{2}\Big) = f\Big(\frac{\pi}{2}\Big)$$

4. For any fixed complex number $z_0 \in \mathbb{C}$, there exists at least one continuous function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\sum_{n\in\mathbb{N}} c_n = z_0.$$

5. If $c_n = 0$ for all $n \in \mathbb{Z}$ such that $|n| > 100$, then f is differentiable.

Solution: We have the formula

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}.$$

Since f is a real-values function, we have $f(x) = \bar{f}(x)$ for all x, thus we deduce

$$\sum_{n\in\mathbb{Z}} c_n e^{inx} = f(x) = \bar{f}(x) = \sum_{n\in\mathbb{Z}} \bar{c}_n e^{-inx} = \sum_{n\in\mathbb{Z}} \bar{c}_{-n} e^{inx} \implies c_n = \bar{c}_{-n}$$

1. The answer is true since c_0 is the average of f.
2. The answer is false. For example, if $c_1 = i, c_{-1} = -i$ and $c_n = 0$ for all $n \neq \pm 1$ then all the coefficients are purely imaginary. But the function f is $-2\sin(x)$ which is odd.
3. The answer is false. For example if $c_{-3} = i, c_3 = -i$ and $c_n = 0$ for all $n \neq \pm 3$, then $f(x) = -2\sin(3x)$ which does not satisfy $f(-\pi/2) = f(\pi/2)$.
4. The answer is false. Indeed $c_n + c_{-n} = c_n + \bar{c}_n \in \mathbb{R}$ and therefore $\sum_n c_n \in \mathbb{R}$ (so if $z_0 = i$ there is no function f).
5. The answer is true. Indeed, the function f is a finite linear combination of the functions e^{inx} for $n \in \{-99, -98, -97, \dots, 98, 99\}$. Since each of such function is smooth, the function f is smooth and in particular differentiable.

Part 2 [10 points]

For each of the following 5 questions, you have to provide a *numeric or symbolic answer*. An example of a numeric answer is $-27\sqrt{2}$; an example of a symbolic answer is a^3/b. Each correct answer is worth 2 points. An empty or wrong answer is worth 0 points.

Insert your answers in the following grid. Write clearly so that there is no ambiguity. Only the answers in the grid will be taken into consideration for grading. You do **not** have to provide any justification for your answers.

Question	1	2	3	4	5
Answer					

Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function such that its Fourier transform equals

$$\hat{f}(\xi) = \frac{e^{-(a+i\xi)}}{a + i\xi}$$

where $a > 0$ is a positive parameter, understood to be unspecified throughout the exercise.

1. As a function of $a > 0$, what is the value of $\int_{-\infty}^{+\infty} f(x)\, dx$?

2. As a function of $a > 0$, what is the value of $\int_{-\infty}^{+\infty} x f(x)\, dx$?

3. As a function of $a > 0$, what is the Fourier transform of the function $x \mapsto f(x + 5)$?

4. In the case when $a = 1$ what is the Fourier transform of the convolution $f * f$?

5. In the case when $a = -3$ what is the value of $\mathcal{F}(f')(1)$?

Solution: We will use the formulas stated in `https://metaphor.ethz.ch/x/2021/hs/401-0373-00L/additional/fourier_formulas.pdf`.

1. We have

$$\int_{\mathbb{R}} f(x)dx = \hat{f}(0) = \frac{e^{-a}}{a}.$$

2. We have

$$\int_{\mathbb{R}} xf(x)dx = \mathcal{F}(xf)(0) = i\Big(\frac{d}{d\xi}\hat{f}\Big)(0) = i\Big(\frac{e^{-(a+i\xi)}(-i)}{a+i\xi} - i\frac{e^{-(a+i\xi)}}{(a+i\xi)^2}\Big)(\xi = 0)$$
$$= e^{-a}(a^{-1} + a^{-2}).$$

3. We have

$$\mathcal{F}(f(x+5))(\xi) = e^{i5\xi}\hat{f}(\xi) = \frac{e^{-(a-i4\xi)}}{a+i\xi}.$$

4. We have

$$\mathcal{F}(f * f) = \hat{f}^2 \frac{e^{-2(a+i\xi)}}{(a+i\xi)^2}.$$

5. We have

$$\mathcal{F}(f') = (i\xi\hat{f}(\xi))(\xi = 1) = \frac{ie^{3-i}}{i-3}.$$

Part 3 [20 points]

Let $u : [0,2] \times [0,\infty) \to \mathbb{R}$ be the solution of the following PDE:

$$\begin{cases} u_t - 5u_{xx} = 0 & \text{for } x \in (0,2),\ t > 0, \\ u(0,t) = 3 & \text{for } t > 0, \\ u(2,t) = 7 & \text{for } t > 0, \\ u(x,0) = 2\left(x + \frac{3}{2}\right) - \frac{\sin(\pi x)}{2} - 4\sin(\pi x)\cos(\pi x) & \text{for } x \in (0,2). \end{cases}$$

(i) Find a function $v(x,t) = v(x)$ which depends only on the x variable that solves

$$\begin{cases} v_t - 5v_{xx} = 0 & \text{for } x \in (0,2),\ t > 0, \\ v(0,t) = 3 & \text{for } t > 0, \\ v(2,t) = 7 & \text{for } t > 0. \end{cases}$$

(ii) Write down the PDE (and its initial and boundary conditions) solved by $w := v - u$ (where u is the original PDE and v is the function constructed in the previous step).

(iii) Compute the Fourier series *in sines only* of the function $\frac{\sin(\pi x)}{2} + 4\sin(\pi x)\cos(\pi x)$.

(iv) With the separation of variables method, find an explicit formula for the function w and deduce from it an explicit expression for the function u.

Solution:

(i) The function v satisfies

$$\begin{cases} -5v'' = 0 & \text{in } (0,2), \\ v(0) = 3, \\ v(2) = 7; \end{cases}$$

hence the function v is affine and is precisely $v(x) = 3 + 2x$.

(ii) The function w satisfies

$$\begin{cases} w_t - 5w_{xx} = 0 & \text{for } x \in (0,2), t > 0, \\ w(0,t) = 0 & \text{for } t > 0, \\ w(2,t) = 0 & \text{for } t > 0, \\ w(x,0) = v(x) - u(x,0) = \frac{\sin(\pi x)}{2} + 4\sin(\pi x)\cos(\pi x) & \text{for } x \in (0,2). \end{cases}$$

(iii) By using the duplication formula for the sine $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$, we get

$$\frac{\sin(\pi x)}{2} + 4\sin(\pi x)\cos(\pi x) = \frac{1}{2}\sin(\pi x) + 2\sin(2\pi x).$$

(iv) Let us find solutions of the equation (up to the initial condition) which are of the form $w(x,t) = X(x)T(t)$. The pde becomes

$$\frac{T'}{T} = 5\frac{X''}{X}$$

and thus, since the two sides depend on different variables, both sides are constant and equal to $-5\lambda^2$. Thanks to the boundary condition, we have

$$\begin{cases} X'' = -\lambda^2 X & \text{in } (0,2), \\ X(0) = 0, \\ X(2) = 0. \end{cases}$$

Thus λ is $\frac{n\pi}{2}$ for a positive integer n and $X(x) = \sin(n\pi x/2)$ up to multiplicative constants. Then, we deduce that $T(t) = e^{-5\pi^2 n^2 t/4}$. Hence, by the superposition principle, we get the general formula for w:

$$w(x,t) = \sum_{n\geq 1} a_n \sin(n\pi x/2) e^{-5\pi^2 n^2 t/4}.$$

By using the Fourier decomposition of $u(x,0)$ found in the previous step, we get that $a_2 = \frac{1}{2}$, $a_4 = 2$, and $a_n = 0$ for all other values of n. Therefore we have

$$w(x,t) = \frac{1}{2}\sin(\pi x)e^{-5\pi^2 t} + 2\sin(2\pi x)e^{-20\pi^2 t}$$

and thus

$$u(x,t) = v(x) - w(x,t) = 3 + 2x - \frac{1}{2}\sin(\pi x)e^{-5\pi^2 t} - 2\sin(2\pi x)e^{-20\pi^2 t}.$$

Part 4 [20 points]

Let $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be the solution of the following PDE:

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) & \text{for } x \in \mathbb{R},\ t > 0, \\ u(x,0) = \begin{cases} \frac{1}{1+x} & \text{for } x \geq 0 \\ \frac{1}{1-x} & \text{otherwise} \end{cases}, \\ u_t(x,0) = \begin{cases} x^2 & \text{for } x \in [-1,2] \\ 0 & \text{otherwise} \end{cases}. \end{cases}$$

(i) Compute $u(3,2)$.

(ii) For any fixed $x \in \mathbb{R}$, what is the value of $\lim_{t\to\infty} u(x,t)$?

(iii) Draw a qualitative graph for the function $x \mapsto u(x,4)$.

(iv) Is the function u continuous?

Solution:

(i) Thanks to the d'Alembert formula, we have

$$u(x,t) = \frac{1}{2}[u(x-t,0) + u(x+t,0)] + \frac{1}{2}\int_{x-t}^{x+t} u_t(\xi,0)d\xi.$$

In particular we get

$$u(3,2) = \frac{1}{2}(\frac{1}{2} + \frac{1}{6}) + \frac{1}{2}\int_1^2 \xi^2 d\xi = \frac{5}{3}.$$

(ii) Notice that $x - t \to -\infty$ and $x + t \to +\infty$ as $t \to \infty$. Moreover, notice that $u(x,0) \to 0$ as $x \to +\infty$ and as $x \to -\infty$. Therefore, by the d'Alembert formula, we get

$$\begin{aligned} \lim_{t\to\infty} u(x,t) &= \frac{1}{2}\lim_{t\to\infty} u(x-t,0) + \frac{1}{2}\lim_{t\to\infty} u(x+t,0) + \frac{1}{2}\lim_{t\to\infty}\int_{x-t}^{x+t} u_t(\xi,0)d\xi \\ &= 0 + 0 + \frac{1}{2}\int_{-\infty}^{+\infty} u_t(\xi,0)d\xi = \frac{1}{2}\int_{-1}^{2} \xi^2 d\xi = \frac{3}{2}. \end{aligned}$$

(iii) For $u(x,4)$ we have the formula (as a consequence of the d'Alembert formula)

$$u(x,4) = \frac{1}{2}\Big[\frac{1}{1+|x-4|} + \frac{1}{1+|x+4|}\Big] + \frac{1}{2}\int_{[x-4,x+4]\cap[-1,2]} \xi^2 d\xi.$$

The first term goes to 0 as $|x| \to \infty$ and is convex in the intervals $(-\infty, -4)$, $(-4, 4)$, $(4, +\infty)$. This is sufficient to draw a qualitative graph of the first term. Now let us focus on the integral:

$$\frac{1}{2}\int_{[x-4,x+4]\cap[-1,2]} \xi^2 d\xi = \begin{cases} 0 & \text{for } x \leq -5, \\ \frac{(x+4)^3+1}{6} & \text{for } -5 \leq x \leq -2, \\ \frac{3}{2} & \text{for } -2 \leq x \leq 3, \\ \frac{8-(x-4)^3}{6} & \text{for } 3 \leq x \leq 6, \\ 0 & \text{for } 6 \leq x. \end{cases}$$

This is sufficient to draw a qualitative graph (in each interval the function is simple).

(iv) Let us recall the d'Alembert formula:

$$u(x,t) = \frac{1}{2}[u(x-t,0) + u(x+t,0)] + \frac{1}{2}\int_{x-t}^{x+t} u_t(\xi,0)d\xi.$$

Since $x \mapsto u(x,0)$ is continuous, the first part is clearly continuous; hence u is continuous if and only if

$$(x,t) \mapsto \int_{x-t}^{x+t} u_t(\xi,0)d\xi = \int_{[x-t,x+t]} u_t(\xi,0)d\xi$$

is continuous. Take two space-time points (x,t) and (x',t'). Given two sets E, F the symmetric difference is given by $E \triangle F := E \setminus F \cup F \setminus E$. We have

$$|u(x,t) - u(x',t')| \leq \int_{[x-t,x+t]\triangle[x'-t',x'+t']} |u_t(\xi,0)|d\xi.$$

Now, notice that $|u_t(\xi,0)| \leq 4$ for all $\xi \in \mathbb{R}$, thus we get

$$|u(x,t) - u(x',t')| \leq 4\Big|[x-t,x+t]\triangle[x'-t',x'+t']\Big|.$$

The continuity of u follows since when $(x',t') \to (x,t)$ the right-hand side goes to 0.