QUICK REVIEW ON THE LAPLACE EQUATION FOR DISK AND ANNULUS

1. Separation of Variables and general approach

First of all notice that we are going to solve the Laplace equation by using polar coordinates (check Problem Set 2 for the calculation) thus we get

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \tag{1}$$

where we have $u(r, \theta)$. Again we are using the Ansatz of separation of variables

$$u(r, \theta) = R(r)\Theta(\theta) \tag{2}$$

Using (2) and plugging this into (1) we get the following equation

$$\Delta u = R''(r)\Theta(\theta) + r^{-1}R'(r)\Theta(\theta) + r^{-2}R(r)\Theta''(\theta) = 0$$

Thus we get the following systems of ODEs in $R(r)$ and $\Theta(\theta)$ that we need to solve

$$\begin{cases} \Theta(\theta)'' = \alpha\Theta(\theta) \text{ (ODE 1)} \\ r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} + \alpha = 0 \text{ (ODE 2)} \end{cases}$$

Now observe that we need periodic solutions in θ and (by basic ODE theory) this can only happen for $\alpha < 0$ and when $\alpha = 0$ (but definitely not for $\alpha > 0$). In the case $\alpha < 0$ we get

$$\Theta(\theta) = A\cos(\sigma\theta) + B\sin(\sigma\theta)$$

where we conveniently wrote $\alpha = -\sigma^2$; since the period of $\Theta(\theta)$ is given by $\frac{2\pi}{\sigma}$ and since this must be a divisor of 2π we conclude that $\sigma = n \in \{1, 2, 3, \dots\}$. Thus we get a family of solutions for ODE 1 of the form:

$$\Theta_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta).$$

When $\alpha = 0$ the only periodic solutions of ODE 1 are of the form

$$\Theta_n(\theta) = A_0.$$

Thus, since $\cos(0) = 1$ and $\sin(0) = 0$ we can conveniently group the solutions for $\alpha \leq 0$ as

$$\Theta_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta), \quad \text{for} \quad n = 0, 1, 2, 3, \dots.$$

For the second ODE using $\alpha = -n^2$ we get the following ODE:

$$r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} - n^2 = 0 \tag{3}$$

$$\overset{\text{ODE theory}}{\Longrightarrow} \begin{cases} R_n \in \langle r^n, r^{-n}\rangle_{\mathbb{R}} & \text{if } n > 1 \\ R_0 \in \langle 1, \ln(r)\rangle_{\mathbb{R}} & \text{if } n = 0 \end{cases} \tag{4}$$

Note that we have **not** spoken about any boundary conditions yet. However we are going to focus on disks and annuli (see Farlow Lesson 33 and Lesson 34, respectively). In the following two subsection we take a look on how to apply this approach in some illustrative examples.

1.1. **The disk.** In the following subsection we are going to one problem concerning a disc.

Example 1. Let $\Omega = D(1) = \{(x, y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$ and consider the following problem:

$$\begin{cases} \Delta u = 0 & \text{in } D(1) \\ u(1,\theta) = 1 + \sin\theta + \frac{3}{2}\sin 2\theta + 2\cos 4\theta & \text{on } \partial D(1) = S^1(1) \end{cases}$$

Step 1: Choosing the right solutions in the general approach

By considering the domain of a disk $D(1)$ and since we want to obtain a **smooth solution** to our problem we need to 'throw away' the unbounded solutions of (**??**). Thus, after using the approach of separation of variable as discussed above, we get the same solutions for Θ_n, however for R_n we **only choose** the following solutions:

$$\begin{cases} R_0(r) = C_0 \text{ for } C_0 \in \mathbb{R} \\ R_n(r) = C_n r^n \text{ for } C_n \in \mathbb{R}, n \geq 1 \end{cases}$$

Putting everything together we get

$$\begin{aligned} u(r,\theta) =& \sum_{n\geq 1} C_n r^n \cdot (A_n \cos(n\theta) + B_n \sin(n\theta)) \\ \overset{\text{rename constants}}{=}& \frac{a_0}{2} + \sum_{n\geq 1} r^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) \end{aligned}$$

Step 2: Using the boundary condition

Now using the boundary condition we get that

$$\begin{aligned} u(1,\theta) =& \frac{a_0}{2} + \sum_{n\geq 1} 1^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) \\ \overset{!}{=}& 1 + \sin\theta + \frac{3}{2}\sin(2\theta) + 2\cos 4\theta \end{aligned}$$

We see directly that the boundary condition **is already in the form of a Fourier Series**. Thus we conclude, just by visual inspection, that $a_0 = 2$, $a_4 = 2$, $b_1 = 1$ and $b_2 = \frac{3}{2}$ where all the other coefficients are equal to 0. We conclude that our final solution is

$$u(r,\theta) = 1 + r\sin(\theta) + \frac{3}{2}r^2\sin(2\theta) + 2r^4\cos(4\theta)$$

1.2. **The annulus.** Here we are going to provide two examples of solving the Laplace Equation on the annulus.

Example 2. Consider $\Omega = \{(x, y) \in \mathbb{R}^2 \mid 1 < \sqrt{x^2+y^2} < 2\}$ and the following problem

$$\begin{cases} \Delta u = 0 & \text{on } \Omega \\ u(1,\theta) = 0 & \text{on } \partial D(1) = S^1(1) \\ u(2,\theta) = 4\sin\theta & \text{on } \partial D(2) = S^1(2) \end{cases}$$

Step 1: Choosing the right solutions from general approach

By the general approach we again get the same solutions for Θ_n and R_n. However since the

origin is excluded from our domain we **can't throw away** the unbounded solutions of (3) as we did in the case of the subsection above. Thus we get

$$u(r,\theta) = A_0C_0 + D_0\ln(r) + \sum_{n\geq 1}\left(C_nr^n + \frac{D_n}{r^n}\right)[A_n\cos n\theta + B_n \sin n\theta]$$

$$\overset{\text{renaming constants}}{=} c_0 + d_0\ln r + \sum_{n\geq 1} r^n\left(a_n\cos(n\theta) + c_n\sin(\theta n)\right) + r^{-n}\left(b_n\cos(n\theta) + d_n\sin(n\theta)\right)$$

$$\overset{\text{renaming constants}}{=} c_0 + d_0\ln r + \sum_{n\in\mathbb{Z}\setminus\{0\}} r^n\left(a_n\cos n\theta + b_n\sin n\theta\right)$$

Using our first boundary condition we see that

$$c_0 + d_0\ln 1 + \sum_{n\in\mathbb{Z}\setminus\{0\}} 1^n\left(a_n\cos(n\theta) + b_n\sin(\theta n)\right) \overset{!}{=} 0$$

thus we see that $c_0 = 0, a_n = -a_{-n}$ and $b_n = b_{-n}$ for all $n = 1, 2, 3, \ldots$; the second boundary condition gives

$$c_0 + d_0\ln 2 + \sum_{n\in\mathbb{Z}\setminus\{0\}} 2^n\left(a_n\cos(n\theta) + b_n\sin(n\theta)\right) \overset{!}{=} 4.\sin(\theta)$$

Hence, we directly conclude that $c_0 = d_0 = 0$ and $a_n = a_{-n} = 0$ for all $n = 1, 2, 3, \ldots$, moreover we have $b_n = b_{-n} = 0$ for $n \neq 1$ thus the remaining part of the equation is

$$\sin(\theta)\left(2b_1 - \frac{1}{2}b_{-1}\right) \overset{b_1=b_{-1}}{=} \sin(\theta)\left(2b_1 - \frac{1}{2}b_1\right) \overset{!}{=} 4\sin\theta$$

this directly gives that $b_{-1} = b_1 = \frac{8}{3}$ and we conclude that the solution is

$$u(r,\theta) = \frac{8}{3}\left(r - \frac{1}{r}\right)\sin\theta$$

Remark. The important takeaway is in Step 1, where it was important to notice that **in difference to the example of the disc** we **don't throw away the unbounded solutions**.

Example 3. Consider the following problem for $\Omega = \{(x,y)\in\mathbb{R}^2 \mid 1 < \sqrt{x^2+y^2} < 2\}$:

$$\begin{cases} \Delta u = 0 & \text{on } \Omega \\ u(1,\theta) = 2\sin\theta & \partial D(1) = S^1(1) \\ u(2,\theta) = 3\sin\theta & \text{on } \partial D(2) = S^1(2) \end{cases}$$

Step 1: Choosing the right solutions from general approach

Similiar to the previous example we get

$$u(r,\theta) = c_0 + d_0\ln r + \sum_{n\in Z\setminus\{0\}} r^n\left(a_n\cos(n\theta) + b_n\sin(n\theta)\right)$$

Step 2: Solve using the boundary conditions

The two boundary conditions impose

$$c_0 + \sum_{n\in\mathbb{Z}\setminus\{0\}} 1^n\left(a_n\cos(n\theta) + b_n\sin(n\theta)\right) \overset{!}{=} 2\sin(\theta)$$

and

$$c_0 + d_0 \ln 2 + \sum_{n \in \mathbb{Z} \setminus \{0\}} 2^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) \stackrel{!}{=} 3 \sin(\theta).$$

Thus, exactly as above, $c_0 = d_0 = 0$ and $a_n = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$ and $b_n = 0$ for $n \in \mathbb{Z} \setminus \{1, -1\}$. The condition to determine b_1, b_{-1} is the 2×2 system:

$$\begin{cases} b_{-1} \sin(-\theta) + b_1 \sin(\theta) = \sin(\theta)\left(b_1 - b_{-1}\right) \stackrel{!}{=} 2 \sin(\theta) \\ \frac{1}{2} b_{-1} \sin(-\theta) + 2 b_1 \sin(\theta) = \sin(\theta)\left(\frac{b_{-1}}{2} + 2 b_1\right) \stackrel{!}{=} 3 \sin(\theta) \end{cases}$$

whence we conclude that $b_{-1} = -\frac{2}{3}$ and $b_1 = \frac{4}{3}$ and so our final solution is

$$u(r, \theta) = \sin(\theta)\left(\frac{4}{3} r + \frac{2}{3r}\right).$$