Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich
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Mathematik III
Final Exam

Identifying number: 001

- Put your student identity card on the desk. **Do not open this booklet** until you are so instructed.
- The exam is pseudonymised: this means that you do **not** have to write your name or your student number on the exam, but you are identified by the number we told you prior to the exam. Please check the correctness of the identifying number above before you start working!
- During the exam no written aids nor calculators or any other electronic device are allowed. **Phones must be switched off and stowed away** in your bag during the whole duration of the exam.
- A4-paper is provided. No other paper is allowed. Write with blue or black pens. **Do *not* use pencils, erasable pens, red or green ink, nor Tipp-Ex.**
- Write your solution to each exercise in the designated white space in the booklet. Leave enough ($\approx 2\,\mathrm{cm}$) empty space on the margins (top, bottom and sides). If you need more space to write your solutions you can use the provided A4-paper, but please start every part on a new sheet of paper and **write your identifying number on every sheet of paper**.
- Please write neatly! Please do not put the graders in the unpleasant situation of being incapable of reading your solutions, as this will certainly not play in your favour!
- In part 3 and part 4 of this exam your answers do need to be properly justified. It is fine and allowed to use theorems/statements proved in class or in the homework (i. e. in Problem Sets 1–12) without reproving them (**unless otherwise stated**), but you should provide a precise statement of the result in question.
- Said $0 \leq s \leq 60$ your total score on the exercises, your final grade in the exam will be *bounded from below* by $\frac{s}{10}$.
- The duration of the exam is **120 minutes**.

Do not fill out this table!

part	points	check
1	[10]	
2	[10]	
3	[20]	
4	[20]	
total	[60]	

grade:

Part 1 [10 points]

For each of the following 5 statement you have to establish whether it is true or false. Each correct answer is worth 2 points. An empty answer is worth 0 points. A wrong answer detracts 2 points (that is, it is worth -2 points). If your total score in this exercise is negative, you will anyway be given 0 points (so, it is impossible to get a negative score on the whole exercise).

Insert your answers in the following grid. Write clearly **T** if the statement is true and **F** if the statement is false. We will accept also **R** if the statement is *richtig* (which is the German word for *true*).

Only the answers in the grid will be taken into consideration for grading.

Question	1	2	3	4	5
Answer					

Let $f : [0, \pi) \to \mathbb{R}$ be given by $f(x) = x - 3\cos(5x)$. Let then $g : (-\pi, \pi) \to \mathbb{R}$ denote the even extension of f, and let us consider its Fourier series

$$S_g(x) = \frac{a_0}{2} + \sum_{n \geq 1} (a_n \cos(nx) + b_n \sin(nx)).$$

1. $a_0 > 0$

2. $b_{90} < 0$

3. $S_g(0) = -3/2$

4. $a_5 = -\frac{1}{5\pi} - 3$

5.

$$\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$$

Hint: Compute $S_g(\pi)$.

Solution:

The string of answers is as follows: **T - F - F - F - T**

Since the function is even $b_n = 0$ for all $n \geq 1$. Let us focus on the Fourier series of x (we denote the coefficient of $\cos(nx)$ with a'_n), since the Fourier series of $-3\cos(5x)$ is gotten by visual inspection. For any $n \geq 1$, a'_n satisfies

$$a'_n = \frac{2}{\pi}\int_0^{\pi} \cos(ns)s\,ds = \frac{2}{\pi n^2}\int_0^{\pi n} \cos(t)t\,dt = \frac{-2}{\pi n^2}\int_0^{\pi n} \sin(t)\,dt = \frac{2}{\pi n^2}((-1)^n - 1).$$

So, for $n \geq 1$ even $a'_n = 0$ and for $n \geq 1$ odd $a'_n = \frac{-4}{\pi n^2}$. For a'_0, we have

$$a'_0 = \frac{2}{\pi}\int_0^{\pi} s\,ds = \pi.$$

Thus, we get that $a_0 = \pi$, $a_n = 0$ if $n = 2, 4, 6, 8, \ldots$, $a_n = \frac{-4}{\pi n^2}$ if n is odd and $n \neq 5$, $a_5 = \frac{-4}{25\pi} - 3$.

Let $h(x) = g(x) + 3\cos(5x)$, that is $h(x) = |x|$ on $[-\pi, \pi]$. Thanks to what we observed above we have

$$\pi = h(\pi) = S_h(\pi) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n \text{ odd}} \frac{1}{n^2} \implies \sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

Part 2 [10 points]

For each of the following 5 questions, you have to provide a *numeric or symbolic answer*. An example of a numeric answer is $-27\sqrt{2}$; an example of a symbolic answer is a^3/b. Each correct answer is worth 2 points. An empty or wrong answer is worth 0 points.

Insert your answers in the following grid. Write clearly so that there is no ambiguity. Only the answers in the grid will be taken into consideration for grading. You do **not** have to provide any justification for your answers.

Note: in the following questions, to be considered correct your answer needs to be explicit, i.e. it should not be given in the form of an unsolved integral or series.

Question	Answer
1	
2	
3	
4	
5	

Throughout this exercise, we let $f(x) = e^{-x^2}$.

Note: We recall that the Fourier transform of $h(x) = e^{-x^2/2}$ was computed in class and is given by $\hat{h}(\xi) = \sqrt{2\pi}e^{-\xi^2/2}$. In answering the following questions we advise you to exploit the properties of the Fourier transform whenever possible.

1. What is the Fourier transform of f?

2. What is the Fourier transform of g, where $g(x) = f(x-1)$?

3. What is the convolution product $f * f$?

4. What is the value of $\int_{\mathbb{R}} x^4 f(x)\,dx$?

5. What is the value $u(x=0, t=1/200)$ where u is the (only integrable) solution to the heat equation

$$\begin{cases} u_t = 100u_{xx} & \text{for } x \in \mathbb{R},\ t > 0, \\ u(x,0) = f(x) & \text{for } x \in \mathbb{R}. \end{cases} \quad ?$$

Solution: To solve this exercise we will use the formula for the Fourier transform of the Gaussian

$$\mathcal{F}(e^{-x^2/2}) = \sqrt{2\pi}e^{-\xi^2/2}. \tag{1}$$

1. Thanks to the scaling properties of the Fourier transform, namely the fact that

$$\phi(x) := \psi(ax) \ \Rightarrow\ \hat{\phi}(x) = a^{-1}\hat{\psi}(\xi/a)$$

from (1) we deduce (since $f(x) = h(\sqrt{2}x)$ and so $a = \sqrt{2}$ in the above)

$$\mathcal{F}(f) = \sqrt{\pi}e^{-\frac{\xi^2}{4}}.$$

2. Thanks to the translation property of the Fourier transform, namely the fact that

$$\phi(x) := \psi(x-a) \ \Rightarrow\ \hat{\phi}(x) = e^{-ia\xi}\hat{\psi}(\xi)$$

we then get (since in our case $a = 1$)

$$\mathcal{F}(g) = \sqrt{\pi}e^{-\left(\frac{\xi^2}{4}+i\xi\right)}$$

3. For the convolution product, we have

$$\mathcal{F}^{-1}(\mathcal{F}(f * f)) = \mathcal{F}^{-1}(\hat{f}\cdot\hat{f}) = \pi\mathcal{F}^{-1}(e^{\frac{-\xi^2}{2}}) = \sqrt{\frac{\pi}{2}}e^{-\frac{x^2}{2}}.$$

4. For the fourth moment, we have

$$\int_{\mathbb{R}} x^4 f(x)\,dx = (-i)^4 \mathcal{F}(x^4 f)(0) = \left(\frac{d}{d\xi}\right)^4 \hat{f}(0) = \frac{\sqrt{\pi}}{16} e^{-\xi^2/4}(12 - 12\xi^2 + \xi^4)(\xi = 0) = \frac{3}{4}\sqrt{\pi}.$$

5. For the last question, as discussed in class the solution is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi c^2 t}} \int e^{-\frac{(x-y)^2}{4c^2 t}} f(y)\,dy$$

hence (notice that $4c^2t = 2$ when $t = 1/200$)

$$u(0, 1/200) = \frac{1}{\sqrt{2\pi}} \int e^{-y^2/2 - y^2}\,dy = \frac{1}{\sqrt{2\pi}} \int e^{-3y^2/2}\,dy = \frac{1}{\sqrt{2\pi}} \int e^{-z^2}\,dz \sqrt{\frac{2}{3}} = \frac{1}{\sqrt{3}}.$$

Part 3 [20 points]

Consider the disc $D := \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 \leq 4\}$, and let $u \in C^2(D; \mathbb{R})$ solve the elliptic boundary value problem

$$\begin{cases} \Delta u = 0 \text{ in } D \\ u = g \text{ in } \partial D \end{cases}$$

where $g(x, y) = -1 + x^2 + 2y$.

1. Write g in polar coordinates (r, θ);
2. Determine $u(0, 0)$, i.e. the value of the solution in question at the center of the disc;
3. Determine $\max_{(x,y)\in D} u(x, y)$ and $\min_{(x,y)\in D} u(x, y)$ and, for each of them, the point(s) where they are attained;
4. Compute the (real) Fourier series expansion of $g(2, \theta)$;
5. Determine the explicit expression of u as a series of functions.

Hint: Recall the trigonometric identity $\cos(2\theta) = 2\cos^2(\theta) - 1$.

Solution:

1. Since g is defined only on $\partial D = \{(x, y) : x^2 + y^2 = 4\}$, we have $x = 2\cos\theta$, $y = 2\sin\theta$. Therefore, $g(r, \theta) = -1 + 4\cos^2(\theta) + 4\sin(\theta)$.
2. The value of $u(0, 0)$ is computed using Poisson's formula as the average of g along the boundary circle (of radius 2); if one employs the identity $\cos(2\theta) = 2\cos^2(\theta) - 1$ then the integration is straightforward:
$$u(0,0) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(-1 + 2(1 + \cos(2\theta)) + 4\sin(\theta))\, d\theta = \frac{1}{2\pi}(-2\pi + 4\pi + 0) = 1.$$
3. Since u is harmonic, the maximum principle guarantees that the maximum and the minimum are attained on the boundary of the domain. So, we have to find the maximum and minimum of $g(2, \theta)$ for $\theta \in [-\pi, \pi]$. In order to find the maximum and minimum of $g(2, \theta)$ (which is a 2π-periodic function), let us find the values of θ where its derivative vanishes
$$0 = \frac{d}{d\theta} g(2, \theta) = -8\cos(\theta)\sin(\theta) + 4\cos(\theta) = -8\cos(\theta)\Big(\sin(\theta) - \frac{1}{2}\Big).$$
Hence, we have that the extremals are attained for some of the values $\theta \in \{\frac{\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}\}$. The corresponding values of $g(2, \theta)$ are $3, -5, 4, 4$. Thus, we conclude
$$\max_{(x,y)\in D} u(x, y) = 4, \text{ attained at } (x, y) = (\sqrt{3}, 1), \text{ and } (x, y) = (-\sqrt{3}, 1)$$
$$\min_{(x,y)\in D} u(x, y) = -5, \text{ attained at } (x, y) = (0, -2).$$

4. Employing the identity $\cos(2\theta) = 2\cos^2(\theta) - 1$ allows to rewrite the function $g(2, \theta)$ as

$$g(2, \theta) = -1 + 2(1 + \cos(2\theta)) + 4\sin(\theta) = 1 + 2\cos(2\theta) + 4\sin(\theta).$$

Hence, if we consider the general form a Fourier series expansion of a 2π periodic function

$$g(2, \theta) = \frac{a'_0}{2} + \sum_{n\geq 1}(a'_n \cos(n\theta) + b'_n \sin(n\theta))$$

we get at once that $a'_0 = 2$ (average term), and $a'_2 = 2$ and $a'_n = 0$ for $n \neq 0, 2$, while $b'_1 = 4$ and $b'_n = 0$ for $n \neq 1$.

5. We have seen in class (Lecture 12) that the general form of an harmonic function on a disc is

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n\geq 1}(a_n \cos(n\theta) + b_n \sin(n\theta))r^n$$

and the coefficients are determined by the boundary value by matching the expansion at $r = 2$ (i.e. imposing the boundary value). Hence,

$$a_n = a'_n/2^n, \quad b_n = b'_n/2^n$$

and the only non-zero coefficients are in fact

$$a_0 = 2,\ a_2 = 1/2,\ b_1 = 2,$$

so the solution takes the form

$$u(r, \theta) = 1 + \frac{1}{2}\cos(2\theta)r^2 + 2\sin(\theta)r.$$

Part 4 **[20 points]**

Consider the initial value problem

$$\begin{cases} u_{tt} = c^2 u_{xx} & \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}_{\geq 0}, \\ u(x,0) = \max\{4 - |x-2|, 0\} & \text{for } x \in \mathbb{R}, \\ u_t(x,0) = 0 & \text{for } x \in \mathbb{R}. \end{cases}$$

where c is a real, positive constant.

1. Determine $u(x,t)$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}_{\geq 0}$;
2. Draw, as a function of $c > 0$, the graph of the value $u(1,1)$;
3. Determine the points $(x,t) \in \mathbb{R} \times \mathbb{R}_{\geq 0}$ where the amplitude of the wave, i.e. the value of u, is maximum, and determine such a value;
4. Compute the value of $\lim_{t \to +\infty} u(-3,t)$;
5. Suppose we now perform a modification of the initial datum $u(x,0)$ only for $x > 0$ (while keeping it unchanged for $x \leq 0$). What is the largest set of points $(x,t) \in \mathbb{R} \times \mathbb{R}_{>0}$ for which the value of $u(x,t)$ may be affected by such a modification? Describe it by means of equations (in terms of c), and draw it. Also: for what values of c will such modifications potentially affect $u(-1,3)$?

Solution:

1. Using d'Alembert's formula, we have

$$\begin{aligned} u(x,t) &= \frac{1}{2}\Big(u(x-ct,0) + u(x+ct,0)\Big) \\ &= \frac{1}{2}\Big(\max\{4 - |x-ct-2|, 0\} + \max\{4 - |x+ct-2|, 0\}\Big). \end{aligned}$$

2. Given $c > 0$, let $\psi(c)$ be the corresponding value of $u(1,1)$. The formula obtained in the previous step implies

$$\psi(c) = \frac{1}{2}\Big(\max(3-c, 0) + \max(4 - |c-1|, 0)\Big).$$

   Let us consider 4 cases depending on the value of $c > 0$:

   - If $0 < c \leq 1$, then the formula simplifies to

$$\psi(c) = \frac{1}{2}\Big(3 - c + 3 + c\Big) = 3.$$

   - If $1 \leq c \leq 3$, them the formula simplifies to

$$\psi(c) = \frac{1}{2}\Big(3 - c + 5 - c\Big) = 4 - c.$$

- If $3 \leq c \leq 5$, the the formula simplifies to

$$\psi(c) = \frac{1}{2}\Big(0 + 5 - c\Big) = \frac{5-c}{2}.$$

- If $5 \leq c$, the the formula simplifies to

$$\psi(c) = \frac{1}{2}\Big(0 + 0\Big) = 0.$$

The plot is then piecewise linear (affine), corresponding to the four cases above.

3. Let $f(z) = \max\{4 - |z-2|, 0\}$. Notice that $0 \leq f(z) \leq 4$ for all possible value of z and $f(z) = 4$ if and only if $z = 2$. Since $u(x,t) = \frac{1}{2}(f(x-ct) + f(x+ct))$ we deduce that $u(x,t) \leq 4$ and $u(x,t) = 4$ if and only if $x - ct = 2$ and $x + ct = 2$. The two conditions imply $x = 2, t = 0$; thus the maximum of u is $u(2,0) = 4$.

4. The formula obtained in the first step tells us

$$u(-3,t) = \frac{1}{2}\Big(\max\{4 - |ct+5|, 0\} + \max\{4 - |ct-5|, 0\}\Big).$$

If $t > 9/c$, then $ct + 5 > 4$ and $ct - 5 > 4$, thus we deduce $u(-3,t) = 0$. In particular, this implies

$$\lim_{t\to+\infty} u(-3,t) = 0.$$

5. The value of $u(x,t)$ depends only on $u(x-ct,0)$ and $u(x+ct,0)$. So, it can change when we perturb $u(z,0)$ for $z > 0$ if and only if $x - ct > 0$ or $x + ct > 0$. Since $x - ct < x + ct$, then $u(x,t)$ can change if and only if $x + ct > 0$.

   In particular, the point $(-1,3)$ can be affected if and only if $-1 + 3c > 0$, that is equivalent to $c > \frac{1}{3}$.