1. Classification of PDEs Classify each of the following 2nd order PDEs according to their homogeneity (homogeneous or nonhomogeneous), linearity (linear or non linear), coefficients (constant or non constant) and, when the PDE is linear, type (parabolic, hyperbolic or elliptic).

Note: An equation may have different types in different regions of the domain of the function.

- $(1+y^2)u_{xx} + e^{-\frac{x^2}{2}}u_{yy} - xu_x + (3-x^2)u_y = 0, \ \ (x,y) \in \mathbb{R}^2;$
- $u_{xx} + 4xy + e^y u_y = (x+y)^2 u_x, \ \ (x,y) \in \mathbb{R}^2;$
- $u_t - u_{xx} - 5u_{tx} = 0, \ \ (t,x) \in \mathbb{R}_{>0} \times \mathbb{R}.$

2. Linear combination of solutions Consider the following PDEs:

$$u_x(x,t) + e^{-2t}u_{xt}(x,t) = F(x,t), \tag{1}$$
$$u_x(x,t) + e^{-2t}u_{xt}(x,t) = 0, \tag{2}$$

and suppose that $v(x,t)$ satisfies (1) and $w(x,t)$ satisfies (2).

1. Is $v(x,t) + w(x,t)$ a solution of (1)?
2. Given $\alpha, \beta \in \mathbb{R}$, is $\alpha v(x,t) + \beta w(x,t)$ a solution of (1)?
3. Given $\beta \in \mathbb{R}$, is $v(x,t) + \beta w(x,t)$ a solution of (1)?
4. Given $\alpha, \beta \in \mathbb{R}$, is $\alpha v(x,t) + \beta w(x,t)$ a solution of

$$u_x(x,t) + e^{-2t}u_{xt}(x,t) = \alpha F(x,t) \ ?$$

3. Symmetries of solutions Let $v(x,t)$ be a solution of the following PDE

$$u_{xx}(x,t) + u_{tt}(x,t) = 0. \tag{3}$$

Determine which of the following three functions solve (3): $v(x,-t)$, $v(-x,t)$, $v(-x,-t)$.

4. Even and odd functions Recall that an odd function is a function such that $f(x) = -f(-x)$ and an even function is a function such that $f(x) = f(-x)$.

1. Describe all functions which are both even and odd.
2. Let f, g be two odd functions. What can we say about $f \cdot g$?
3. Let f, g be two even functions. What can we say about $f \cdot g$?

4. Let f be an odd function and let g be an even function. What can we say about $f \cdot g$?
5. Let f, g be two odd functions. What can we say about $f + g$?
6. Let f, g be two even functions. What can we say about $f + g$?
7. Let f be an odd function and let g be an even function. What can we say about $f + g$?
8. Let f, g be odd functions. What can we say about $(f + g)^2$?
9. For which values of $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$, the function $\sin^n(\alpha x)$ is odd? For which values it is even?

5. Computation of the real Fourier series For each of the following function $\phi : [-L, L] \to \mathbb{R}$, find its real Fourier series, i.e., find the coefficients $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 1}$ such that

$$\phi(x) = a_0 + \sum_{n\geq 1} a_n \cos(\frac{n\pi}{L}x) + b_n \sin(\frac{n\pi}{L}x).$$

1. $L = 2$ and
$$\phi(x) = \begin{cases} 1 & \text{for } x \in (-1, 2), \\ 0 & \text{for } x \in (-2, -1). \end{cases}$$

2. $L = \frac{1}{2}$ and
$$\phi(x) = \begin{cases} \sin(2\pi x) & \text{for } x \in (0, \frac{1}{2}), \\ 0 & \text{for } x \in (-\frac{1}{2}, 0). \end{cases}$$

6. Computation of the cosine Fourier series For each of the following function $\phi : [0, L] \to \mathbb{R}$, find the cosine Fourier series of its even extension, i.e., find the coefficients $(a_n)_{n\geq 0}$ such that for each $0 \leq x < L$ one has

$$\phi(x) = \sum_{n\geq 0} a_n \cos(\frac{n\pi}{L}x).$$

1. $L = \pi$ and
$$\phi(x) = \begin{cases} 0 & \text{for } x \in (0, \frac{\pi}{2}), \\ 2 & \text{for } x \in (\frac{\pi}{2}, \pi). \end{cases}$$

2. $L = 2$ and
$$\phi(x) = x^2.$$

7. True or false on Fourier series For each of the following 5 statements you have to establish whether it is true or false.

Insert your answers in the following grid. Write clearly **T** if the statement is true and **F** if the statement is false. We will accept also **R** if the statement is *richtig* (which is the German word for *true*).

Question	1	2	3	4	5
Answer					

Let $f(x)$ and $g(x)$ be two continuous functions defined on the interval $[-L, L]$. Suppose that their Fourier series are respectively

$$a_0 + \sum_{n\geq 1} a_n \cos(\frac{n\pi}{L}x) + b_n \sin(\frac{n\pi}{L}x)$$

and

$$c_0 + \sum_{n\geq 1} c_n \cos(\frac{n\pi}{L}x) + d_n \sin(\frac{n\pi}{L}x).$$

1. The Fourier series of $f + g$ is

$$(a_0 + c_0) + \sum_{n\geq 1}(a_n + c_n) \cos(\frac{n\pi}{L}x) + (b_n + d_n) \sin(\frac{n\pi}{L}x).$$

2. The Fourier series of $f \cdot g$ is

$$(a_0 c_0) + \sum_{n\geq 1}(a_n c_n) \cos(\frac{n\pi}{L}x) + (b_n d_n) \sin(\frac{n\pi}{L}x).$$

3. Any odd continuous function defined on $[-L, L]$ has a Fourier series without the cosine terms.

4. The Fourier series of $\frac{1}{2}(f(x) + f(-x))$ is

$$a_0 + \sum_{n\geq 1} a_n \cos(\frac{n\pi}{L}x).$$

5. The Fourier series of $\frac{1}{2}(f(x) - f(-x))$ is

$$a_0 + \sum_{n\geq 1} a_n \cos(\frac{n\pi}{L}x).$$

8. Double frequence

Consider the function $f(x) = x(\pi - x)$ for $0 \le x \le \pi$ and let g be its even extension to $(-\pi, \pi)$.

1. Sketch a graph of g.
2. Determine the Fourier series of g, i.e. determine all coefficients in the expansion

$$S_g(x) = \frac{a_0}{2} + \sum_{n\ge 1}(a_n \cos(n\theta) + b_n \sin(n\theta)).$$

3. Suitably using the previous expansion, compute the sum $\sum_{n\ge1} \frac{(-1)^n}{n^2}$.

9. Fourier transform and its use

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = (x-1)e^{-2(x+1)^2}$.

1. Compute the Fourier transform of f.
2. Using the previous part of this exercise, compute the value of $\int_{-\infty}^{+\infty}(9+x)f(x)\,dx$.

10. Convolution For each of the following 5 questions, you have to provide a *numeric or symbolic answer*. An example of a numeric answer is $-27\sqrt{2}$; an example of a symbolic answer is a^3/b. Insert your answers in the following grid. Write clearly so that there is no ambiguity.

Note: in the following questions, to be considered correct your answer needs to be explicit, i.e. it should not be given in the form of an unsolved integral or series.

Question	1	2	3	4	5
Answer					

For each of the following pairs of functions f and g, compute the corresponding convolution $f * g$.

1. $f(x) = x^2$, $g(x) = e^{-|x|}$.

2. $f(x) = x^2$, $g(x) = e^{-x^2}$.
3. $f(x) = \sin(x)$, $g(x) = e^{-x^2}$.
4. $f(x) = g(x) = 1$ for $0 \leq x \leq 1$ and 0 otherwise.
5. $f(x) = x$, $g(x) = \frac{1}{1+x^2}$.

11. Fourier transform and ODEs

Let us consider the following ODE on the real line:

$$-u''(x) + u(x) = e^{-2|x|}, \quad x \in \mathbb{R}.$$

1. Show there exists a *unique* solution u such that $u(x) \to 0$ as $x \to \pm\infty$.
2. Determine such a solution. (Namely: determine the only solution of the ODE in question such that $u(x) \to 0$ as $x \to \pm\infty$).

12. Heat equation via separation of variables Consider the PDE given by

$$\begin{cases} u_t(x,t) = 4u_{xx}(x,t) & \text{for } 0 < x < 10,\ t > 0, \\ u(0,t) = u(10,t) = 0 & \text{for } t \geq 0, \\ u(x,0) = \phi(x) & \text{for } 0 \leq x \leq 10. \end{cases}$$

Using the separation of variables method, solve it for the following choices of the initial datum:

1. $\phi(x) = \sin(5\pi x) - 3\sin(\pi x)$,
2. $\phi(x) = 2\sin(\frac{7\pi x}{2})$,
3. $\phi(x) = \sin(3\pi x) + 2\cos(\frac{(6x+5)\pi}{10})$,
4. $\phi(x) = 6\cos^2(\pi x - \frac{\pi}{4}) - 3$.

13. Employing the heat kernel Solve the IVP

$$\begin{cases} u_t - u_{xx} = 0 & (x,t) \in \mathbb{R} \times \mathbb{R}_{>0} \\ u(x) = e^{-2x}. \end{cases}$$

You are required to work out your answer explicitly (i.e. the answer should not be given in the form of an unsolved integral).

14. Wave equation on an interval Find the solution $u : [0, L] \times [0, \infty) \to \mathbb{R}$ of the following PDE:

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) & \text{for } 0 < x < L,\ t > 0, \\ u(0,t) = u(L,t) = 0 & \text{for } t \geq 0, \\ u(x,0) = 0 & \text{for } 0 \leq x \leq L, \\ u_t(x,0) = \sin\left(\frac{8\pi}{L}x\right)\cos\left(\frac{3\pi}{L}x\right) & \text{for } 0 \leq x \leq L. \end{cases}$$

15. Wave profile

Consider the wave IVP given by

$$u_{tt} = 9u_{xx} \ \text{ for } x \in \mathbb{R}, t \geq 0 \qquad u(x,0) = x \qquad u_t(x,0) = 6xe^{-x^2}.$$

1. Compute $u(1,1)$.
2. Determine the points (if any) where the amplitude of the wave vanishes.
3. Determine

$$\sup_{x\in\mathbb{R},t\geq 0} u(x,t), \quad \inf_{x\in\mathbb{R},t\geq 0} u(x,t)$$

and, in either case, the points where such values are attained (if any).

16. Spectra under various BC

Given $L > 0$ consider the square $[0, L] \times [0, L] \subset \mathbb{R}^2$. Compute the spectrum of the Laplacian, i.e. the values $\lambda \in \mathbb{R}$ such that the equation

$$\Delta u = -\lambda u$$

has non-trivial (namely: not identically zero) solutions, under either of the following boundary conditions:

1. Neumann (=normal derivative equal to zero) on two *opposite* sides of the square, and Dirichlet (=value of the function equal to zero) on the other two;
2. Neumann (=normal derivative equal to zero) on two *adjacent* sides of the square, and Dirichlet (=value of the function equal to zero) on the other two.

Then compare the results you got.

17. Dirichlet problem on a disc Let $D := \{(x,y) \in \mathbb{R}^2 \ : \ x^2+y^2<9\}$. Solve the problem

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u(x,y) = 1-3x^2+xy+y^2 & \text{on } \partial D. \end{cases}$$

Compute $u(0,0)$, as well as the maxima and minima of u on D. Then explicitly write down the solution both in polar coordinates (r,θ) and in Cartesian coordinates (x,y).

18. Radially symmetric Poisson problem on a disc

Let $D := \{(x,y) \in \mathbb{R}^2 \ : \ x^2+y^2<1\}$. Consider the problem

$$\begin{cases} \Delta u = (x^2+y^2)^2 & \text{in } D \\ u(x,y) = 0 & \text{on } \partial D. \end{cases}$$

1. Show this problem has a *unique* solution.
2. Determine the solution in question.

Note: in the following two problems we employ the spherical coordinates (r,φ,θ) as introduced in Lecture 14.

19. Interior Dirichlet problem

Solve the interior Dirichlet problem, in the unit ball of $\mathbb{R}^3$, given by

$$\begin{cases} \Delta u = 0 \\ u(1,\varphi) = \cos(3\varphi). \end{cases}$$

Hint: firstly, write $\cos(3\varphi)$ in terms of $\cos(\varphi), \cos^2(\varphi), \cos^3(\varphi)$; secondly, write

$$\cos(3\varphi) = a_0P_0(\cos(\varphi)) + a_1P_1(\cos(\varphi)) + a_2P_2(\cos(\varphi)) + a_3P_3(\cos(\varphi))$$

for suitable a_0, a_1, a_2, a_3 to be determined. Then proceed as in Lecture 14.

20. Thermic dipole

Consider a metal ball of radius 1, whose boundary temperature is fixed to equal

$$g(\varphi) = \begin{cases} +1 & \text{if } 0 \leq \varphi \leq \pi/2 \\ -1 & \text{if } \pi/2 \leq \varphi \leq \pi. \end{cases}$$

Explicitly determine, to third order, the steady state temperature near the center of the ball, i.e. determine $v(r, \varphi)$ such that $u(r, \varphi) = v(r, \varphi) + O(r^4)$ as $r \to 0$ where u is the steady state temperature.