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Thursday, September 30, 2021 12:29 AM

1.1. Linear ODE with constant coefficients. Solve (i.e. determine the set of all solutions of) the following differential equations for $y(x)$:

(a) $y'' - \omega^2 y = 0$,

(b) $y'' + \omega^2 y = 0$,

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1.2. First-order ODE with variable coefficients. Solve (i.e. determine the set of all solutions of) the following differential equations for $y(x)$:

(a) $y' - x^2 y = 0$, $x \in \mathbb{R}$,

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(c) $y' + x^5 y = x^6 + 1$, $x \in \mathbb{R}$,

(d) $y' = (x + y)^2$,

(e) $y' - y = \sin x$,

(f) $y\,y' - (1 + y)x^2 = 0$.

Tips: ODE of 1st order may be solved by *separation of variables* or by substitution. For (c), multiply the equation with $e^{f(x)}$, where f is a suitable function. For (f), y will not explicitly be a function of x. It is enough to write a relation between the function y and the variable x that does not contain any derivatives of y.

1.3. Initial and boundary value problems. Solve the following Cauchy problems:

(a) $\begin{cases} y' = 2e^{2x} & \forall x \in \mathbb{R}, \\ y(0) = 2. \end{cases}$

(b) $\begin{cases} y''(x) + 4y(x) = 0 & \forall x \in (0, L)\ (L > 0 \text{ given}), \\ y(0) = 0, \\ y(L) = 2. \end{cases}$

1.4. Spring pendulum A spring pendulum consists of a coil spring and a mass test piece (with mass m) attached to it, which can move in a straight line in the direction in which the spring extends or retracts. Let $K > 0$ be the spring constant and $\omega^2 := K/m$, then the equation of motion of the spring pendulum is given by

$$\ddot{x}(t) + \omega^2 x(t) = 0. \tag{1}$$

Find the solution of the differential equation (1):

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HG F 28

Hand-in for people registered in my class.

(a) with the initial conditions $x(0) = 1, \dot{x}(0) = 2\omega$.

(b) with the boundary conditions $x(0) = 1, x(\frac{\pi}{2\omega}) = 1$.

1.5. Classification of PDEs I. Suppose a, b, f and g are differentiable functions. Tell whether the following differential equations in $u(x, y)$ are linear and homogeneous, linear and inhomogeneous, or non-linear and (in any case) tell their order. For every linear differential equation of 2nd order, tell whether the equation is elliptic, hyperbolic or parabolic.

(a) $u_{xxx} + u_y = f$

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(d) $2u_{xx} + u_x + 2u_{xy} + 2u_{yy} = 0$

(e) $(1 - x^2)u_{xx} - 2xyu_{xy} + (1 - y^2)u_{yy} = g$ auf $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 > 1\}$.

1.6. Classification of PDEs II.

Suppose a, b and g differentiable functions with $g > 0$.

Tell whether the following differential equations in $u(x, y)$ are linear and homogeneous, linear and inhomogeneous, or non-linear and tell their order. For every linear differential equation of 2nd order, tell whether the equation is elliptic, hyperbolic or parabolic.

(a) $au_{xxx} + b(u^4 + u) = 0$

(b) $a^2 u_{xx} + u_x u_y = 1$

(c) $4u_{xx} + u_x + u_{xy} + 6u_{yy} = 0$

(d) $(x^2 - 2)u_{xx} + 4xyu_{xy} + (y^2 - 2)u_{yy} = g$ auf $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 > 16\}$.

1.7. Dreaming a Cauchy-Lipschitz theorem for the wave equation. (*Achtung: for this problem we don't expect you to write down anything; this is instead about thinking ahead.*) Read (the first three pages of) **Lesson 16** in Farlow's textbook, about the 'derivation' of the (one-dimensional) wave equation, as describing a vibrating string. Compare this equation with Newton's equation, as recalled in class. What 'data' would you expect one has to specify for such wave equation for something like the Cauchy-Lipschitz theorem (i.e. local existence and uniqueness) to hold true? We will discuss this at length in the coming lectures.

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(1) $a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c\, y(x) = \boxed{0}$

Constant $a, b, c \in \mathbb{R}$, $a > 0$

Space of solutions: 2^{nd} order $\Rightarrow$ 2 independent solution

Say $y_1(x)$, $y_2(x)$ solve (1) and are independent

then $c_1 y_1(x) + c_2 y_2(x) = 0 \iff c_1, c_2 = 0$

If u solves (1) then u can be written as sum of y_1 and y_2

In most cases look for:

$\{ C_\lambda e^{\lambda x} \mid C_\lambda$ constant, λ solves the characteristic equation of (1) $\}$

$\lambda \in \mathbb{C}, \mathbb{R}$

Plug in $C_\lambda e^{\lambda x}$:

$$\underbrace{(a\lambda^2 + b\lambda + c)}_{\text{characteristic equation}} C_\lambda e^{\lambda x} = 0$$

$\Rightarrow$ 2nd polynomial.

$$D = b^2 - 4ac$$

$$\lambda_\pm = \frac{-b \pm \sqrt{D}}{2a}$$

Constants C_λ are usually found from the boundary values.

General sol: $C_{\lambda_+} e^{\lambda_+ x} + C_{\lambda_-} e^{\lambda_- x}$

Bound val: $y(0) = 1$

$y'(0) = 1$

$D = b^2 - 4ac$

$$\lambda_{\pm} = \frac{-b \pm \sqrt{D}}{2a}$$

3 Possibilities:

Textbook:

1) $b^2 - 4ac > 0$ $\longrightarrow$ $u(x) = C_{\lambda_+} e^{\lambda_+ x} + C_{\lambda_-} e^{\lambda_- x}$ is the solution.

$$u(x) = e^{-\frac{b}{2a}x}\left(C_{\lambda_+} e^{+\sqrt{D}x} + C_{\lambda_-} e^{-\sqrt{D}x}\right)$$

$$\frac{e^{\sqrt{D}x} \pm e^{-\sqrt{D}x}}{2} = \cosh(\sqrt{D}x) \text{ or } \sinh(\sqrt{D}x)$$

2) $b^2 - 4ac < 0$

$\sqrt{D}$ is imaginary $\longrightarrow$ Try solution $e^{-\frac{b}{2a}x}\left(C_+ \cos(\sqrt{D}x) + C_+ \sin(\sqrt{D}x)\right)$

why?

Euler tells us: $e^{i\gamma x} = \cos(\gamma x) + i \sin(\gamma x)$ $\sqrt{-1} = i$

C_+ C_-

$$e^{-\frac{b}{2a}x}\left(C_+ e^{i\sqrt{D}x} + C_{\lambda_-} e^{-i\sqrt{D}x}\right)$$

$D = -4$

$\sqrt{-4} = i2$

$\lambda_{\pm} = \pm i$

$y''(x) = -y(x)$

$\lambda^2 + 1 = 0$ $\lambda = \pm i$ $y(0) = 1$

$y(x) = \cos(x)$

3) Double root. $D = 0$ $\lambda_\pm = -\frac{b}{2a}$ is the same

$\underbrace{C_{\lambda_+} e^{\lambda x} + C_{\lambda_-} e^{\lambda x}}$ → not lin independent!

Textbook: try $C_1 e^{\lambda x} + C_2 \underline{x e^{\lambda x}}$ ($+ C_3 x^2 e^{\lambda x}$) 3rd ord

Why? $f(x) = C_1 e^{\lambda x}$

Build 2nd solution out of $g(x) = C_2 e^{\lambda x} v(x)$

↳ Plug in:

Plug in $\quad q(x) = C_2 e^{\lambda x} v(x)$

$$0 = a\left(v''(x) - \frac{b}{a}v'(x) + \frac{b^2}{4a^2}\right)e^{-\frac{b}{2a}x}\cancel{C_2} + b\left(v'(x) - \frac{b}{2a}v(x)\right)\cancel{e^{-\frac{b}{2a}x}}\cancel{C_2} + c\,v(x)\,\cancel{e^{-\frac{b}{2a}x}}\cancel{C_2}$$

$$= \left(a v''(x)\right) + \left(\left(-b + b\right)v'(x)\right) + \left(\frac{b^2}{4a} - \frac{b^2}{2a} + c\right)v(x)$$

$$= a v''(x) - \frac{1}{4a}\underbrace{\left(b^2 - 4ac\right)}_{D=0} v(x) = 0$$

$$0 = \cancel{a}\, v''(x) \qquad v(x) = \gamma x + \beta$$

$$C_1 e^{-\frac{b}{2a}x} + (\gamma x + \beta)\, C_2\, e^{-\frac{b}{2a}x}$$

βC_2

absorb

$$C_1 e^{-\frac{b}{2a}x} + C_2\, x\, e^{-\frac{b}{2a}x}.$$

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$$a(x)\cancel{\frac{d^2y}{dx^2}} + b(x)\frac{dy}{dx} + c(x)y(x) = 0$$

Separation of variables + substitution

$$x^2 + 2x\,y(x) + y(x)^2$$

$$y'(x) = (x+y)^2$$

$$z(x) = x + y(x)$$

$$z'(x) = 1 + y'(x)$$

$$z'(x) - 1 = y'(x)$$

$$z'(x) \cancel{-1} = z(x)^2 + 1$$

$$\frac{dz}{dx} = z^2 + 1$$

$$\frac{dz}{z^2+1} = \cancel{z^2+1}\, dx$$

$$\frac{d}{dz}\arctan(z) = \frac{1}{1+z^2}$$

$$\int \frac{dz}{z^2+1} = \int dx$$

add constant

$$\arctan(z) = x + C$$

$$z = \tan(x + C) \rightarrow y(x) = \tan(x+C) - x$$

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$\boxed{\textbf{(e)}\ y' - y = \sin x,}$

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$e^{x^6/6}\, y(x)$

$$\int u(x)\, v'(x)\, dx = u(x)\, v(x) - \int u'(x)\, v(x)\, dx$$

Integration by parts.

$$y'(x) - y = \sin(x)$$

$$e^{-x} y'(x) - e^{-x} y = \sin(x)\, e^{-x}$$

$$\frac{d\left(e^{-x} y(x)\right)}{dx} = \sin(x)\, e^{-x}$$

$$e^{-x} y(x) = \int \sin(x)\, e^{-x}\, dx$$

$$= \frac{1}{2}\left(\cos(x) - \sin(x)\right)$$

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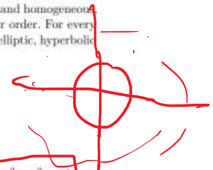
A B C



$$A^{(x,y)}\, \partial_x^2 u + B^{(x,y)}\, \partial_x \partial_y u + C\, \partial_y^2 u + D\, \partial_x u + E\, \partial_y u + F = G$$

u_{xx}

if coefficients vary ~~can vary~~ then can vary over domain

1) Parabolic : $B^2 - 4AC = 0$

2) Hyperbolic : $B^2 - 4AC > 0$

3) Elliptic : $B^2 - 4AC < 0$

Why only A, B, C? $\rightarrow$ 2nd der. determine behaviour of the solution the most

$$(4xy)^2 - 4(1-x^2)(1-y^2) = -4 + 4(x^2+y^2) > 0$$

>1

hyperbolic

Determinant of matrix?

$$\vec{\nabla} = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}$$

$\vec{\nabla}^T \vec{A}\, \vec{\nabla} u$ $\qquad \begin{pmatrix} D \\ E \end{pmatrix} \cdot \vec{\nabla} u$

F

$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}^T \cdot \vec{A} \cdot \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} u + \vec{b}\, \vec{\nabla} u + C\, u(x)$$

$$\begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix}$$

$$\det(-\vec{A}) = \frac{B^2}{4} - AC.$$

$$\cancel{A} \quad A\,\partial_x\partial_x + \frac{B}{2}\,\partial_x\partial_y + \frac{B}{2}\,\partial_y\partial_x + C\,\partial_y\partial_y$$

$$(\xi_1, \xi_2) \in \mathbb{R}^2 \longrightarrow \text{symbol of the PDE}$$

$$f(\xi_1, \xi_2) = A\,\xi_1^2 + \overset{C}{\cancel{B}}\,\xi_2^2 + B\,\xi_1\,\xi_2 \neq 0$$

y^

$$A = 8\ ,\ C = 8$$

Thursday, September 30, 2021 10:21 AM

$A = 8, \quad C = 8, \quad B = 1$

8 x^2 + 8 y^2 + xy

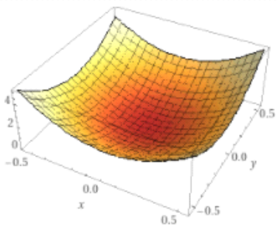
NATURAL LANGUAGE | MATH INPUT | EXTENDED KEYBOARD | EXAMPLES | UPLOAD | RANDOM

Input

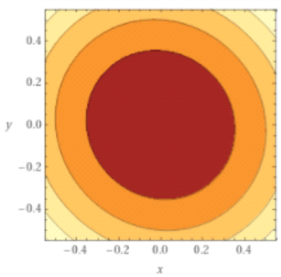
$8x^2 + 8y^2 + xy$

3D plot

Show contour lines



Contour plot

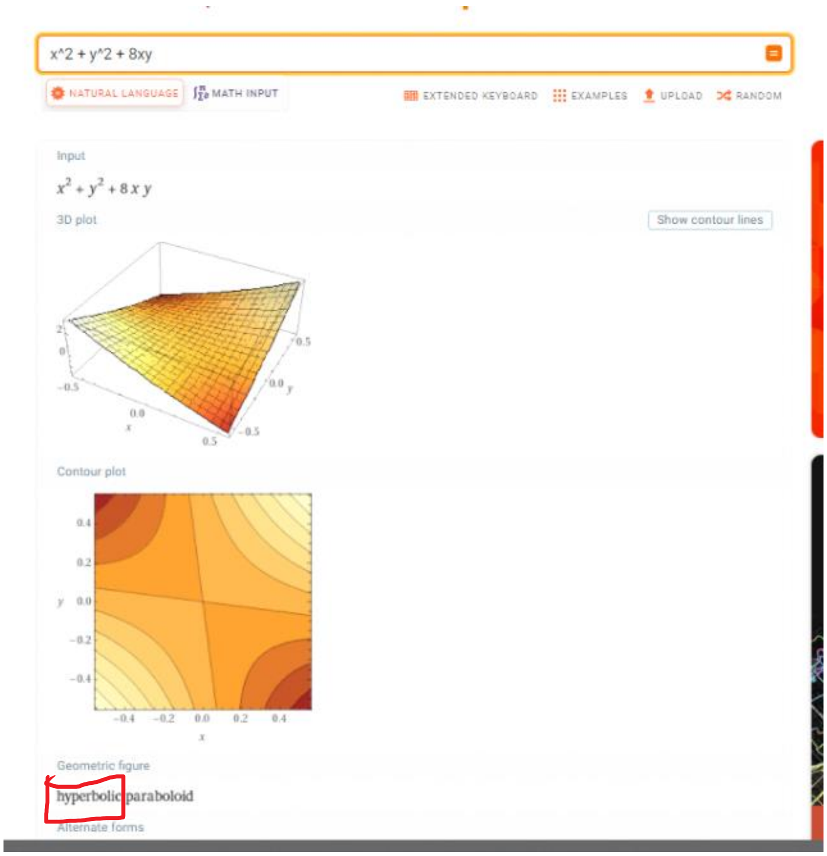


Geometric figure

$\boxed{\text{elliptic}}$ paraboloid

Alternate forms

More


x^2 + y^2 + 8xy
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3D plot
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Contour plot
Geometric figure
hyperbolic paraboloid
Alternate forms

$A = 1, C = 1, B = 8$