Thursday, December 2, 2021 12:56 AM

Consider the following IVP:

$$\begin{cases} \boxed{u_{tt}=u_{xx}} & \text{for } x\in\mathbb{R} \text{ and } t>0,\\ u(x,0)=f(x) & \text{for } x\in\mathbb{R},\\ u_t(x,0)=g(x) & \text{for } x\in\mathbb{R},\ t>0.\end{cases}$$

For each of the following change of coordinates, write the corresponding IVP for $v(\xi,\eta)=u(\xi(x,t),\eta(x,t))$.

1. $\xi=4x, \eta=5t$.
2. $\boxed{\xi=x+3t, \eta=t.}$
3. $\xi=2x, \eta=xt$.

Use the chain rule:

$$\boxed{\xi(x,t)=x+3t\ ,\ \eta(x,t)=t}$$

Note: $\frac{\partial \xi}{\partial x}=1,\ \frac{\partial \xi}{\partial t}=3,\ \frac{\partial \eta}{\partial x}=0,\ \frac{\partial \eta}{\partial t}=1$

Be patient and apply chain rule

up to the order of 2-nd derivatives!

$u(\xi(x,t),\eta(x,t))$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\overbrace{\frac{\partial \xi}{\partial x}}^{1} + \frac{\partial u}{\partial \eta}\overbrace{\frac{\partial \eta}{\partial x}}^{=0} = \frac{\partial u}{\partial \xi}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\overbrace{\frac{\partial \xi}{\partial t}}^{3} + \frac{\partial u}{\partial \eta}\overbrace{\frac{\partial \eta}{\partial t}}^{=1} = 3\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2}\overbrace{\frac{\partial \xi}{\partial x}}^{=1} + \cancel{\frac{\partial^2 u}{\partial \eta\partial \xi}\frac{\partial \eta}{\partial x}}^{=0} = \frac{\partial^2 u}{\partial \xi^2}$$

$$\frac{\partial^2 u}{\partial t^2} = 3\frac{\partial^2 u}{\partial \xi^2}\underbrace{\frac{\partial \xi}{\partial t}}_{3} + 3\frac{\partial^2 u}{\partial \eta\partial \xi}\underbrace{\frac{\partial \eta}{\partial t}}_{1} + \frac{\partial^2 u}{\partial \xi\partial \eta}\underbrace{\frac{\partial \xi}{\partial t}}_{3} + \frac{\partial^2 u}{\partial \eta^2}\underbrace{\frac{\partial \eta}{\partial t}}_{1}$$

$$= 9\frac{\partial^2 u}{\partial \xi^2} + 6\frac{\partial^2 u}{\partial \eta\partial \xi} + \frac{\partial^2 u}{\partial \eta^2}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial t}\right)\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial t}\right)\frac{\partial \eta}{\partial t}$$

$$\frac{\partial u}{\partial t}(\xi(x,t), \eta(x,t))$$

Putting it all back together:

$$\begin{cases} \cancel{9}\ 8\frac{\partial^2 u}{\partial \xi^2} + 6\frac{\partial^2 u}{\partial \eta\partial \xi} + \frac{\partial^2 u}{\partial \eta^2} = \cancel{\frac{\partial^2 u}{\partial \xi^2}}\ 0 \\ u(\xi(x,0),0) = f(\xi(x,0)) \quad (\text{ut } t=0 \quad \xi(x,0)=x) \\ 3\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} = g(\xi(x,0)) \end{cases}$$

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10.3. Vanishing mixed derivative

Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a twice differentiable function such that $u_{xy} = 0$ vanishes identically. Then show that there exists (twice differentiable) functions $a, b : \mathbb{R} \to \mathbb{R}$ such that

$$u(x,y) = a(x) + b(y), \quad \text{for all } (x,y) \in \mathbb{R}^2.$$

(This is the reason why it is convenient, e.g. in studying the 1D wave equation, to introduce the *canonical coordinates*).

Lecture 9

$\xi(x,t) = x + ct$

$\eta(x,t) = x - ct$

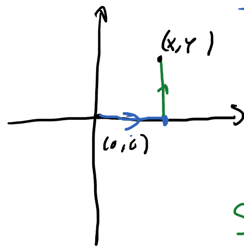
$$\frac{\partial^2 u}{\partial t^2} = c^2 (w_{\xi\xi} + 2 w_{\xi\eta} + w_{\eta\eta})$$

$$\frac{\partial^2 u}{\partial x^2} = w_{\xi\xi} - 2 w_{\xi\eta} + w_{\eta\eta}$$

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}} \iff -4c^2 w_{\xi\eta} = 0 \iff w_{\xi\eta} = 0.$$

$$\Rightarrow w = A(\xi) + B(\eta).$$

(Fact!)



Proof: WLOG take $(x_0, y_0) = (0,0)$

Step 1: Integrate from $x_0 = 0$ to x

(*) $$\underbrace{u_y(x,y_0)}_{f} - \underbrace{u_y(x_0,y_0)}_{f} = \int_{x_0}^{x} \overbrace{\underbrace{u_{xy}}_{\partial_x f}(x',y_0)}^{=0} \, dx' = 0$$

$$u_y(x,y_0) = u_y(x_0,y_0)$$

Step 2: Integrate from $y_0 = 0$ to y.

(*) $$\int_{y_0}^{y} u_y(x,y') \, dy = u(x,y) - u(x,y_0)$$

$$u(x,y) = \underbrace{u(x,y_0)}_{a(x)} + \underbrace{u(x_0,y) - u(x_0,y_0)}_{b(y)}$$

(*) Note: small correction with respect to live session: what I meant to derive was

$$u_y(x,t) - u_y(x_0,t) = \int_{x_0}^{x} \overbrace{u_{xy}}^{=0}(x',t) \, dx' = 0$$

i.e. with $t \in [y_0, y]$ arbitrary but fixed

Then it follows $u_y(x,t) = u_y(x_0,t)$ $\forall t \in [y_0, y]$

This we can integrate on both sides:

$$\int_{y_0}^{y} u_y(x,t) \, dt = \int_{y_0}^{y} u_y(x_0,t) \, dt$$

$$u(x,y) - u(x,y_0) = u(x_0,y) - u_y(x_0,y_0)$$

$$u(x,y) = \underbrace{u(x,y_0)}_{a(x)} + \underbrace{u(x_0,y) - u(x_0,y_0)}_{b(y)}$$

Inhomogeneous d'Alembert: trip down memory lane?

9.2. Inhomogeneous wave equation on the real line. Let $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be a solution of the PDE

$$\begin{cases} u_{tt} - 4u_{xx} = \sin(4t) + x & \text{for } x \in \mathbb{R},\ t > 0, \\ u(x,0) = 2x^2 & \text{for } x \in \mathbb{R}, \\ u_t(x,0) = 6\cos(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

Reminder: Duhamel's principle (3 steps)

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

Step 0: $u = u_1 + u_2$

Solve with Duhamel,

$$\begin{cases} \partial^2_{tt} u_1 - c^2 \partial^2_{xx} u_1 = F(x,t) \\ u_1(x,0) = 0 \\ \partial_t u_1(x,0) = 0 \end{cases}$$

solve with d'Alembert!

$$\begin{cases} \partial^2_{tt} u_2 - c^2 \partial^2_{xx} u_2 = 0 \\ u_2(x,0) = f(x) \\ \partial_t u_2(x,0) = g(x) \end{cases}$$

Step 1: transform retarded ini data.

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0 \\ v(x,s) = 0 \\ v_t(x,s) = F(x,s) \end{cases}$$

auxilliary in (x,t,s)
$u(x,t,s)$ solution

$w(x,t) = v(x, t+s)$

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0 \\ w(x,0) = 0 \\ w_t(x,0) = F(x,s) \end{cases}$$

d'Alembert !!! $\quad w(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} F(u,s)\, du$

$$u(x,t,s) = v(x,t) = w(x, t-s) = \frac{1}{2c} \int_{x-c(t-s)}^{x+c(t-s)} F(u,s)\, du$$

Step 2: (Duhamel) solution for u_1 given by

$$u_1(x,t) = \int_0^t u(x,t,s)\, ds$$

(only remains to check u_1 is a solution!)

$F(x,t) = \sin(4t) + x$

$$u_1(x,t) = \int_0^t \underbrace{u(x,t,s)}\, ds = \int_0^t w(x, t-s)\, ds$$

$$= \frac{1}{4} \int_0^t \int_{x-c(t'-s)}^{x-2(t'-s)} F(x,s)\, ds\, dt'$$

$$= \frac{1}{4} \int_0^t \int_{x-2(t'-s)}^{x+2(t'-s)} x + \sin(4s)\, ds\, dt'$$

$$-\frac{1}{4}\cos(4s)\Big|_{x-2(t'-s)}^{x+2(t'-s)}$$

$$= \boxed{\frac{1}{4} t + 2t^2 x - \frac{1}{4}\sin(4t)}$$

$$u_2(x,t) = \frac{2(x-2t)^2 + 2(x+2t)^2}{2} + \frac{1}{4} \int_{x-2t}^{x+2t} 6\cos(u)\, du$$

(d'Alembert)

$$= \boxed{2x^2 + 8t^2 + 3\cos(x)\sin(2t)}$$

$$u = u_1(x,t) + u_2(x,t)$$

$$= 2x^2 + 8t^2 + \frac{1}{4} t + 2t^2 x - \frac{1}{4}\sin(4t)$$

$$+ 3\cos(x)\sin(2t)$$ ✓

same as last time!