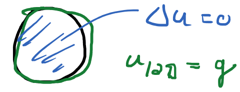


Recap Poisson Kernel

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n\in\mathbb{N}} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) r^n$$

where u solves $\begin{cases} \Delta u = 0 & \text{on } D(a) \\ u = g & \text{on } \partial D(a) = S^1(a) \end{cases}$

$u(a,\theta) = g(\theta)$

Euler: $e^{it} = \cos(t) + i\sin(t)$

$$u(r,\theta) = \sum_{n\in\mathbb{Z}} c_n r^{|n|} e^{-in\theta}$$

with c_n determined by BVP $:\Rightarrow u(a,\theta) = g(\theta)$

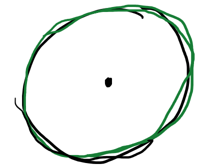
$$c_n = a^{-|n|} \int_0^{2\pi} g(\varphi)\, e^{-in\varphi}\, d\varphi$$

Define u through Poisson kernel

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} g(\varphi)\, P\left(\frac{r}{a}, \theta - \varphi\right) d\varphi \ ,$$

where

$$\boxed{P(q,t) = \underbrace{\frac{1 - q^2}{1 - 2q\cos(t) + q^2}}_{\text{Poisson Kernel!}}}$$



2 Properties

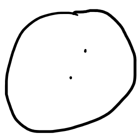
1) Mean value property $\quad r \to 0 \quad P(0, \theta - \varphi) \equiv 1. \quad \Rightarrow$

$$\boxed{u(0,\theta) = \frac{1}{2\pi} \int_0^{2\pi} g(\varphi)\, d\varphi}$$

2) Min-max principle $\quad \min_{\partial\Omega} u \le u(r,\theta) \le \max_{\partial\Omega} u$

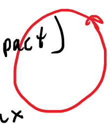
$t = 0 \qquad P(q,0) = \frac{1-q^2}{1+q^2-2q} = \frac{(1+q)(1-q)}{(1-q)(1-q)} = \frac{1+q}{1-q}$

if there is an interior max or min then u is constant.



(Ω is closed & bounded, $\Leftrightarrow$ compact)

$\Rightarrow u$ assumes a min & max



12.1. Laplace equation For each of the following 5 statement you have to establish whether it is true or false.

Insert your answers in the following grid. Write clearly **T** if the statement is true and **F** if the statement is false. We will accept also **R** if the statement is *richtig* (which is the German word for *true*).

Only the answers in the grid will be taken into consideration for grading.

Question	1	2	3	4	5
Answer	F	T	T	T	F

In this exercise, $\Delta = \partial_{xx} + \partial_{yy}$ is the Laplace operator in $\mathbb{R}^2$.

1. If $\Delta u = 0$ and $\Delta v = 0$ for all $(x,y) \in \mathbb{R}^2$, then for any real-valued smooth functions $c_1, c_2 : \mathbb{R}^2 \to \mathbb{R}$, we have
$$\Delta(c_1 u + c_2 v) = 0.$$
2. If $\Delta u = 0$ in $\mathbb{R}^2$, then $\Delta(\partial_x u) = 0$ in $\mathbb{R}^2$.
3. Let $D := \{(x,y) : x^2+y^2 < 4\}$. There exist infinitely many functions $u : D \to \mathbb{R}$ such that
$$\begin{cases} \Delta u = 5 & \text{in } D \\ u(1,1) = 0. \end{cases}$$
4. Let $D := \{(x,y) : x^2+y^2 = 1\}$. If $u : D \to \mathbb{R}$ solves
$$\begin{cases} \Delta u = 0 & \text{in } D, \\ u(x,y) = 2 - x^3 & \text{for } (x,y) \in \partial D, \end{cases}$$
then $u(0, \frac{1}{2}) > 1$.
5. Let $D := \{(x,y) : x^2+y^2 = 1\}$. If $u : D \to \mathbb{R}$ solves
$$\begin{cases} \Delta u = 0 & \text{in } D, \\ u(x,y) = \sin(x) & \text{for } (x,y) \in \partial D, \end{cases}$$
then $u(0,0) = \frac{1}{\pi}$.

$u \in C^\infty$

1.) c_1, c_2 constant?

$\Rightarrow \Delta(c_1 u + c_2 v) = c_1 \Delta u + c_2 \Delta v$ ✓

Counter example:

$u = 1$, $v = 0$.

$c_1(x,y) = x^2$, $c_2(x,y) = 0$

$\Delta(x^2) = 2$

2) $\Delta(\partial_x u) = \partial_x^2 \partial_x u + \partial_y^2 \partial_x u$

$= \partial_x \boxed{\partial_x^2 u} + \partial_x \boxed{\partial_y^2 u}$

$= \partial_x(\partial_x^2 u + \partial_y^2 u)$

$= \partial_x(\underbrace{\Delta u}_{=0}) = 0$ ✓

'u'

General strategy for 3) in constructing infinitely many solutions:

1.) Construct infinitely many solutions $u_1^{(k)}$ that satisfy $\Delta u_1^{(k)} = 0$, $u_1^{(k)}(1,1) = 0$

2) Construct a u_2 that solves the inhomogeneous problem: $\Delta u_2 = 5$

3) Calculate $u_1^{(k)}(1,1) + u_2(1,1) = u_2(1,1)$

4) Substract the constant $u_2(1,1)$ if necesary (this was the function $u_3(x,y) = -u_2(1,1)$ in my example).

5) Final solution: $u^{(k)}(x,y) = u_1^{(k)}(x,y) + u_2(x,y) + u_3(x,y)$
$= u_1^{(k)}(x,y) + u_2(x,y) - u_3(x,y)$

$k \geq 0$ arbitrily

3) $u^{(k)}(x,y) = \boxed{\mathrm{Re}\left(\left((x+iy) - (i+1)\right)^{k}\right)} + \boxed{\frac{5}{2}(x-1)^2} + \cancel{1}$ wrong!

u_2 u_3

$\Delta\left(\mathrm{Re}(\tilde{z}^k)\right) = 0$

$\Delta u_1 = 0$ $\Delta u_3 = 0$ $\Delta u_2 = 5$

for every $k \geq 0$ there is a solution!

4) $\Delta u_1 = 5$ $(1,1)$

$2 - x^3 \geq 2 - 1 \Rightarrow g \geq 1$.

Min/max $\Rightarrow u(0, 1/2) \geq 1$

5) $u(0,0) = \frac{1}{2\pi}\int_0^{2\pi} u(\cos(\theta), (\sin\theta))\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \sin(\cos\theta)\, d\theta = 0$

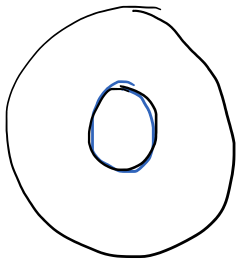
Achtung!!! In our case $u_2(1,1) = 0$ so we don't need to substract. $u_2(1,1)$ of the final solution (I thought $u_2(1,1) = -1$ initially)

12.3. Laplace equation in an annulus Let $D := \{(x,y) : 1 < \sqrt{x^2+y^2} < 2\}$. Find the solution $u : D \to \mathbb{R}$ of the following problem

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ u(1,\theta) = 2\sin(2\theta) & \text{for } 0 \le \theta < 2\pi, \\ u(2,\theta) = 3\cos(3\theta) & \text{for } 0 \le \theta < 2\pi. \end{cases}$$

Visualize D

$-\sin(x) = \sin(-x)$



u is still harmonic on D.

→ We can use its general form

$$\boxed{u(r,\theta) = \sum_{n\in\mathbb{Z}} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) r^n}$$

First boundary condition

$u(1,\theta) = 2\sin(2\theta) \quad 0 \le \theta < 2\pi$

$$\sum_{n\in\mathbb{Z}} a_n \cos(n\theta) + b_n \sin(n\theta) = 2\sin(2\theta)$$

$n=0 \quad a_0 = 0, \ b = 0$

$n \ge 1 \quad a_n + a_{-n} = 0$

$n = 1; n > 2 \quad b_n - b_{-n} = 0$

$b_2 - b_{-2} = 2$

$$\boxed{a_{-3}\cos(3\theta) r^{-3}} + a_{-2}\cos(-2\theta) r^{-2} + \dots$$

$$\dots + a_2 \cos(2\theta) r^2 + \boxed{a_3 \cos(3\theta) r^3}$$

$$= \boxed{3\cos(3\theta)}$$

Second boundary condition

$u(2,\theta) = 3\sin(3\theta)$

$$\sum_{n\in\mathbb{Z}} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) 2^n = 3\cos(3\theta)$$

$a_0 = b_0 = 0$

$2^n b_n - 2^{-n} b_{-n} = 0 \quad \forall n \ge 1$

$2^n a_n + 2^n a_{-n} = 0 \quad$ for $n = 1, 2, \ n > 3$

$8 a_3 + \frac{1}{8} a_{-3} = 3$

$$\begin{cases} 8 a_3 + \frac{1}{8} a_{-3} = 3 \\ a_3 + a_{-3} = 0 \end{cases} \Rightarrow a_3 = \frac{24}{63}, \quad a_{-3} = -\frac{24}{63}$$

$$\begin{cases} b_2 - b_{-2} = 2 \\ 4 b_2 - \frac{1}{4} b_{-2} = 0 \end{cases} \Rightarrow b_2 = \frac{-2}{15}, \quad b_{-2} = \frac{-32}{15}$$

$$u(r,\theta) = \frac{24}{63}(r^3 - r^{-3})\cos(3\theta) + \frac{1}{15}(-2r^2 + 32r^{-2})\sin(2\theta)$$