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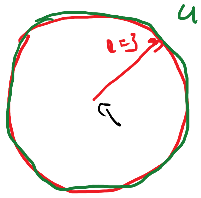
Problem 1. Let $D := \{(x,y) \in \mathbb{R}^2 : \boxed{x^2+y^2<9}\}$. Consider the problem

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u(x,y) = 7 + 5xy + 2y^2 & \text{on } \partial D. \end{cases}$$

(1) Determine $\boxed{u(0,0)}$;

(2) Write down the solution both in polar coordinates (r,θ) and in Cartesian coordinates (x,y).



$u(3,\theta) = \cancel{g(\theta)}$ $f(\theta)$

$\to r=3$

1) Polar coordinates:
$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta). \end{cases}$$

$\Delta u = 0$

$u(3,\theta) = f(\theta) = 7 + 45\boxed{\cos(\theta)\sin(\theta)} + 18\sin^2(\theta)$

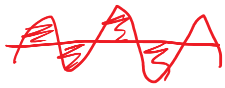
$= 16 + \frac{45}{2}\sin(2\theta) - 9\cos(2\theta)$

$\cos(\theta)\sin(\theta) = \frac{1}{2}\sin(2\theta)$

$\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta))$

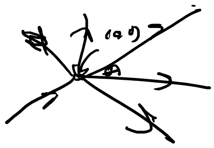
u harmonic $\Rightarrow$ use mean value theorem!

$u(0,0) = \frac{1}{2\pi \cancel{r}}\int_0^{2\pi} f(\theta)\cancel{r}\,d\theta$



$= 16 + \frac{1}{2\pi}\int_0^{2\pi}\underbrace{\cancel{\frac{45}{2}\sin(2\theta)}}_{=0}\,d\theta + \frac{1}{2\pi}\int_0^{2\pi}(-9)\underbrace{\cancel{\cos(2\theta)}}_{=0}\,d\theta = 16$

determine through BC's



2) Remember: general solution $\rightsquigarrow$ $u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n\left(a_n\cos(n\theta) + b_n\sin(n\theta)\right)$

$u(3,\theta) = 16 + \frac{45}{2}\sin(2\theta) - 9\cos(2\theta)$

$\frac{a_0}{2} + \sum_{n=1}^{\infty} 3^n\left(\underbrace{a_n\cos(n\theta)}_{n=2} + \underbrace{b_n\sin(n\theta)}_{n=2}\right) = 16 + \boxed{\frac{45}{2}\sin(2\theta)}\ \boxed{-9\cos(2\theta)}$

$\frac{a_0}{2} = 16$	$3^2 a_2 = -9$	$9 b_n = \frac{45}{2}$
$a_0 = 32$	$a_2 = -1$	$b_n = \frac{5}{2}$

$\cos^2\theta + \sin^2\theta$

$\sin^2(\theta) = \frac{1}{2}(1+\cos(2\theta))$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$\sin(2\theta) = 2\cos\theta\sin\theta$

$u(r,\theta) = 16 + r^2\left(-\cos(2\theta) + \frac{5}{2}\sin(2\theta)\right)$

$x = r\cos\theta$
$y = r\sin\theta$

$u(x,y) = 16 + r^2\left(-(\cos^2\theta - \sin^2\theta) + 5\cos\theta\sin\theta\right)$

transform back

$= 16 + y^2 - x^2 + 5xy$

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(Fourier series can be used to calculate infinite sums) $\longrightarrow$ both are related FT is cont. of Fourier series $\sum_{n\in\mathbb{Z}} \longleftrightarrow \int_{-\infty}$

Fourier transforms can be used to calculate infinite integrals

$$F[f](\xi) = \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx$$

Problem 2. Let $f:\mathbb{R}\to\mathbb{R}$ be given by $f(x) = xe^{-4(x-1)^2}$.

(1) Compute the Fourier transform of f.

(2) Using the previous part of this exercise, compute the value of $\int_{-\infty}^{+\infty}(1+x)f(x)\,dx$.

Know the key identities! (Lecture 6)

a) $F[c_1 f_1 + c_2 f_2](\xi) = c_1 F[f_1](\xi) + c_2 F[f_2](\xi)$

b) $F[x^n f](\xi) = (i\frac{d}{d\xi})^n F[f](\xi)$

c) (Translation $\longleftrightarrow$ Modulation) $a\in\mathbb{R}\setminus\{0\}$, $d\in\mathbb{R}$,

$F[f(a(x-d))] = \frac{1}{a} e^{-i\xi d} F[f](\frac{\xi}{a})$

Also important: $g(x) = e^{-x^2/2}$, $F[g](\xi) = \sqrt{2\pi}\, e^{-\xi^2/2}$ (d)

1) $F[f](\xi) = \int_{\mathbb{R}} x e^{-4(x-1)^2} e^{-ix\xi}\, dx$ $\quad dx \quad dx' \quad x \longleftrightarrow x'+1$

$= \int_{\mathbb{R}} (x'+1) e^{-4x'^2} e^{-i(x'\boxed{+1})\xi} dx'$

$= \boxed{e^{-i\xi}} \underbrace{\int_{\mathbb{R}} x e^{-4x^2} e^{-ix\xi} dx}_{I_1} + \underbrace{\int_{\mathbb{R}} e^{-4x^2} e^{-ix\xi} dx}_{I_2}$

$h(x) = e^{-4x^2}$

$h(x) = g(\sqrt{8}x)$

$a = \sqrt{8}$ Mod. property

$I_2 = F[h](\xi) = \boxed{\frac{1}{\sqrt{8}} F[g](\frac{\xi}{\sqrt{8}})}$ (c) $= \boxed{\frac{\sqrt{2\pi}}{2\sqrt{2}} e^{-\xi^2/16}}$ (d) $= \frac{\sqrt{\pi}}{2} e^{-\xi^2/16}$

$I_1 = \boxed{i\frac{d}{d\xi}(F[h](\xi)) = -\frac{i\xi\sqrt{\pi}}{16} e^{-\xi^2/16}}$ (b)

$F[f](\xi) = e^{-i\xi}(I_1 + I_2) = e^{-i\xi}\left(-\frac{i\xi\sqrt{\pi}}{16} + \frac{\sqrt{\pi}}{2}\right) e^{-\xi^2/16}$

$= \boxed{e^{-i\xi}\left(\sqrt{\pi}\left(\frac{8-i\xi}{16}\right) e^{-\xi^2/16}\right)}$

2) $\int_{\mathbb{R}} (x+1) f(x) e^{-i\xi}\, dx = \boxed{F[x f(x)](\xi=0) + F[f](\xi=0)}$ (a) $\quad \xi = 0$

$= \boxed{i\frac{d}{d\xi}(F[f](\xi=0))}$ (b) $+ F[f](\xi=0)$

$= \boxed{\left[\frac{\sqrt{\pi}}{16}\left(9 - 2i\xi - \frac{\xi^2}{8}\right) e^{-\xi^2/16 - i\xi}\right]_{\xi=0}} + \left[\frac{\sqrt{\pi}}{2} e^{-\xi^2/16}\right]_{\xi=0}$

$= \sqrt{\pi}\,\frac{17}{16}$