(a) Use chain rule to prove

$$\partial_r u(r,\theta) = (\partial_x u)\cos\theta + (\partial_y u)\sin\theta$$
$$\partial_\theta u(r,\theta) = -(\partial_x u) r\sin\theta + (\partial_y u) r\cos\theta.$$

Now we have the following relation for the partial derivatives $\partial_x u$ and $\partial_y u$:

$$\begin{pmatrix} \partial_r u \\ \partial_\theta u \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix}.$$

(b) By inverting some matrix, prove the following expressions for $\partial_x u$ and $\partial_y u$:

$$\partial_x u = \cos\theta(\partial_r u) - \frac{1}{r}\sin\theta(\partial_\theta u),$$
$$\partial_y u = \sin\theta(\partial_r u) + \frac{1}{r}\cos\theta(\partial_\theta u).$$

Use these formulas and chain rule to compute the direct expressions for $\partial^2_{xx} u$ and $\partial^2_{yy} u$ in polar coordinates, i.e.

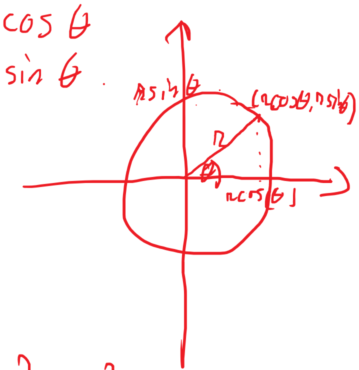
$$\partial^2_{xx} u = \partial_x(\partial_x u) = \cos\theta(\partial_r(\partial_x u)) - \frac{1}{r}\sin\theta(\partial_\theta(\partial_x u)) = \ldots$$

(c) Combine all the information above and prove the following expression for the Laplacian operator in polar coordinates

$$\boxed{\Delta u(r,\theta) = \partial^2_{rr} u + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial^2_{\theta\theta} u.}$$

$u(x,y) \longrightarrow u(r,\theta)$

$x = r\cos\theta$
$y = r\sin\theta$



$\partial_x u \qquad \partial_y u$

$u(x,y) = u(r\cos\theta, r\sin\theta)$

$$\partial_r u(r,\theta) = \frac{\partial u}{\partial x}\underbrace{\frac{\partial}{\partial r}(r\cos\theta)}_{} + \frac{\partial u}{\partial y}\underbrace{\frac{\partial}{\partial r}(r\sin\theta)}_{}$$

$$\frac{\partial u}{\partial x}\frac{\partial x}{\partial r} \qquad \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$$= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\cos\theta.$$

$$\partial_\theta u(r,\theta) = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)$$

$$\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} \qquad \vec{\nabla} u = \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix}$$

$$\det\begin{pmatrix} a & d \\ c & b \end{pmatrix} = ab - cd \neq 0$$

$$\frac{1}{ab-cd}\begin{pmatrix} b & -d \\ -c & a \end{pmatrix}$$

$$\begin{pmatrix} \partial_r u \\ \partial_\theta u \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}\begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix}$$

$$\begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix} = \frac{1}{r}\begin{pmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \partial_r u \\ \partial_\theta u \end{pmatrix}$$

$\cancel{\partial_x}\ \Delta u = \partial_{xx} u + \partial_{yy} u$

substitute   substitute from previous

$\partial_x(\partial_x u) + \partial_y(\partial_y u)$

$\partial_x u = U(r, \theta)$

$\Delta \cancel{u} = \partial^2_{rr} u + \frac{1}{r} \partial_r u + \frac{1}{r^2} \partial^2_\theta u \rightarrow$ indepent of θ.

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$D_v u(\vec{w}) = \frac{d}{dt} u(w + tv)\big|_{t=0}$

2.3. The heat equation on a thin disk Consider a thin (homogeneous) metal disk of radius $r_0 > 0$, whose temperature profile we shall describe in polar coordinates by means of a function $u(r, \theta, t)$. The initial temperature, at time t_0, is a known function $u_0 = u_0(r, \theta)$ and the body is completely insulated.

(a) Write down the full initial boundary value problem (IBVP) modelling the situation described above.

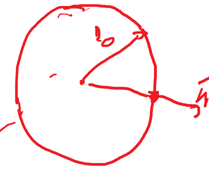
Tip: what is the exterior unit normal to a disk?

(b) What is the solution of this problem in the special case when the initial temperature is constant (equal to T_0)?

(c) Consider the problem in the special case when the initial temperature is a purely radial function, i.e. $u_0(r, \theta) = v_0(r)$. Make the ansatz that also the solution $u(r, \theta, t)$ does not depend on θ, i.e., $u(r, \theta, t) = v(r, t)$. Write down the equations satisfied by v. Prove that the quantity

$$\int_0^{r_0} r\, v(r, t)\, dr$$

does not depend on t. What is the phisical meaning of such quantity? Can you find the asymptotic state of this solution? The asymptotic state is the limit function $v_\infty(r) := \lim_{t \to \infty} v(r, t)$ and can be obtained by coupling the equations satisfied by v with the additional requirement $\partial_t v = 0$.



$u(t, r, \theta)$

$\partial_t u = \Delta u$ → use polar coordinates

$\vec{n}$ → radial direction!

$t_0 = 0$

$u(t_0, r, \theta) = u_0(r, \theta)$

$0 \le r \le r_0$

$0 \le \theta \le 2\pi$

→ independent of t.

$\partial_t u = \partial_{rr}^2 u + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial_{\theta\theta}^2 u$

$t > t_0$

Isolated?

→ Heat bath

Dirichlet or Neumann boundary conditions?

$u(r_0, \theta, t)$?

$\partial_{\vec{n}} u = \partial_r u(r_0, \theta, t)$

$u_0(r, \theta) \equiv T_0$

$u(r, t, \theta) \equiv T_0$ Everywhere.

Every spatial derivative represents heat flow!

$\partial_r u \quad \partial_t u$

$\partial_r u(r_0, t, \theta) = 0$

$\forall\, t \ge t_0,\ 0 \le \theta \le 2\pi$

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$$\int_0^{r_0} \boxed{r\, v(r,t)}\, dr$$

does not depend on t. What is the phisical meaning of such quantity? Can you find the asymptotic state of this solution? The asymptotic state is the limit function $v_\infty(r) := \lim_{t\to\infty} v(r, t)$ and can be obtained by coupling the equations satisfied by v with the additional requirement $\partial_t v = 0$.

$\partial_t u = \Delta u$ has rotational symmetry

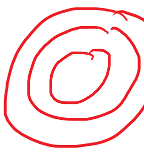
$\partial^2_{\theta\theta}$ vanishes.

$u_0(r, \theta) = v_0(r)$

$u(t, r, \theta) = v(t, r)$

$\partial_t v = \partial^2_{rr} v + \frac{1}{r} \partial_r v$

$\partial_r v(r_0, t) = 0$ for $t > t_0$



$$\frac{1}{\pi r_0^2} \int_0^{2\pi} \int_0^{r} r\, v(r,t)\, dr\, d\theta = \frac{\cancel{2\pi}}{\pi r_0^2} \int_0^{r} r\, v(r,t)\, dr$$

Jacobian

Average heat

$$\frac{d}{dt} \int_0^{r_0} r\, v(r,t)\, dr = \int_0^{r_0} r\, \partial_t v(r,t)\, dr$$

$$= \int_0^{r_0} \underbrace{r\, \partial^2_{rr} v + \partial_r v}_{\text{Chain rule in reverse}}\, dr$$

$$= \int_0^{r_0} \partial_r (r\, \partial_r v)\, dr$$

Primitive is $r\, \partial_r v$

$$= r\, \partial_r v(r,t) \Big|_0^{r_0}$$

Neumann

$$= 0 - 0 = 0$$

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$m, n \in \mathbb{Z}$

Q: what if $a = b$?

$$\int_0^T \sin\Big(\underbrace{2\pi m \frac{t}{T}}_{\frac{2\pi m}{T} = a}\Big) \sin\Big(\underbrace{2\pi n \frac{t}{T}}_{\frac{2\pi n}{T} = b}\Big)\, dt$$

$$\frac{\sin((a-b)t)}{a-b} \qquad \frac{\sin((a+b)t)}{a+b}$$

constantly 1 if $a=b$

$2a$

$$\int_0^T \sin(at)\sin(bt)\,dt = \frac{1}{2}\int_0^T \cos((a-b)t) - \cos((a+b)t)\,dt$$

$$= \frac{1}{2}\left[\frac{\sin((a-b)T)}{a-b} - \frac{\sin((a+b)T)}{a+b}\right]$$

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\cos(\alpha-\beta) = \cos(\alpha)\underbrace{\cos(-\beta)}_{\cos(\beta)} - \sin(\alpha)\underbrace{\sin(-\beta)}_{-\sin(\beta)}$$

use $\frac{1}{2}\left(\cos(\alpha-\beta) + \cos(\alpha+\beta)\right)$

$$\int_0^T \cos(at)\cos(bt)\,dt \overset{?}{=} \frac{1}{2}\int_0^T \cos((a+b)t) + \cos((a-b)t)\,dt$$

$$= \frac{1}{2}\,\frac{\sin((a+b)T)}{a+b} + \frac{\sin((a-b)T)}{a-b}$$

$$\int_0^T \sin(at)\cos(bt)\,dt \overset{?}{=} \frac{1}{2}\int_0^T \sin((a+b)t) - \sin((a-b)t)\,dt$$

$$= \frac{1}{2}\left(-\frac{\cos((a+b)T)}{a+b} + \frac{\cos((a-b)T)}{a-b}\right)$$

use $\frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)$

Always be careful when $a = b$ $\rightsquigarrow$ (i.e. $n = m$)