Thursday, October 14, 2021 8:44 AM

Some facts about even and odd functions

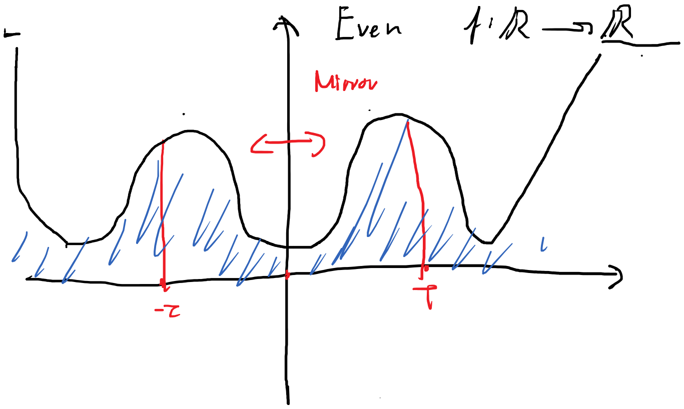
- Even if $f(x) = f(-x)$
- Odd if $-f(x) = f(-x)$

Sam-up load

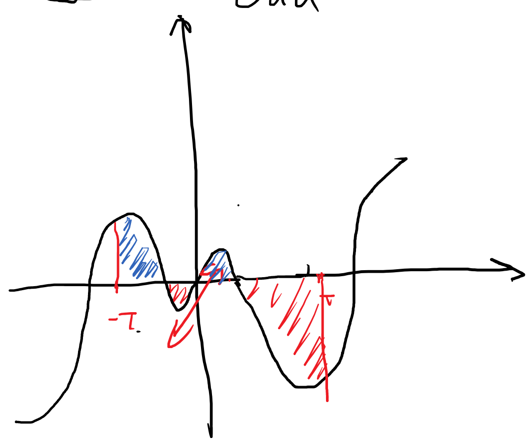
At ETH: Website → Link

At Home: VPN connection

Even $f: \mathbb{R} \to \mathbb{R}$



Odd



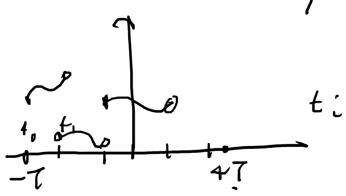
Fun fact: every function can be written as a sum of even and odd function:

$$f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{even}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{odd}}$$

$f_+(x)$ $\quad f_-(x)$

Let now $X = \{ f: [-T, T] \to \mathbb{R} \mid f \text{ is piecewise } C^1 \}$

we can integrate!



If $f \in X$ is even then:

$$\int_{-T}^{T} f(t)\,dt = 2\int_0^T f(t)\,dt$$

$$\int_{-T}^{T} f(t)\,dt = \int_0^T f(t)\,dt + \int_{-T}^{0} f(t)\,dt \qquad t \to -t$$

$$= \int_0^T f(t)\,dt + \int_0^T \underbrace{f(-t)}_{\text{even!} \Rightarrow f(-t) = f(t)}\,dt$$

$$= 2\int_0^T f(t)\,dt$$

f odd $\int_{-T}^{T} f(t)\,dt = \int_0^T f(t)\,dt + \int_{-T}^{0} f(t)\,dt$

$$= \int_0^T f(t)\,dt + \int_0^T f(-t)\,dt$$

$$-f(t)$$

$$= \int_0^T f(t)\,dt - \int_0^T f(t)\,dt = 0$$

$f(x)$	$g(x)$	$f(x)\,g(x)$
E	E	E
O	O	E
E	O	O
O	E	O

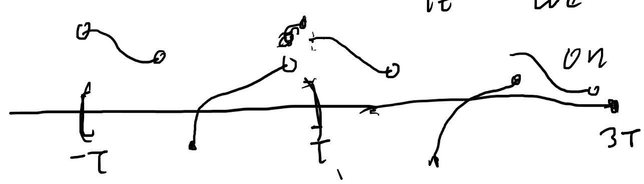
Main idea: Every space of piecewise C^1 Periodic functions has a basis of cos and sin functions.

We know every function can be written as even and odd sum

Attempt to write every periodic even and odd function as a sum of cos, and sin

If that is true it also holds for any interval

If we look at piecewise C^1 !!! on $(-T, T)$ we can repeat them to infinity

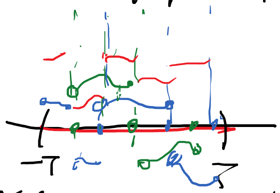


$X = \{ f : [-T,T] \to \mathbb{R} \mid f \text{ piecewise } C^2 \}$ is a vector space. !

- $\{ t_0 = -T, t_1, \dots, t_m = T \}$
- $f|_{[t_i, t_{i+1}]}$ is C^2
- $|f(t)| + |f'(t)| < C$

1) $0 \in X$

2) $f + g \in X$ → choose refinement



3) $\lambda f \in X$

in $\mathbb{R}^3$ every vector is represented by coordinates / "coefficients"

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{\langle x, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + \frac{\langle x, e_2 \rangle}{\langle e_2, e_2 \rangle} e_2 + \frac{\langle x, e_3 \rangle}{\langle e_3, e_3 \rangle} e_3$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

For piecewise C^2, X

Basis given by

$$e_0 = 1, \quad e_n = \frac{1}{\sqrt{T}} \cos\left(\frac{n\pi}{T} t\right), \quad f_n = \frac{1}{\sqrt{T}} \sin\left(\frac{n\pi}{T} t\right)$$

Scalar product is given by $\boxed{\langle f(x), g(x)\rangle = \int_{-T}^{T} f(x)\, g(x)\, dx}$

So any $h \in X$ we can write as

$$h(x) = \underbrace{\frac{\langle e_0, h(x)\rangle}{\langle e_0(x), e_0(x)\rangle}}_{a_0} e_0(x) + \sum_{n=1}^{\infty} \underbrace{\frac{\langle e_n, h(x)\rangle}{\langle e_n, e_n\rangle}}_{a_n} e_n(x) + \sum_{n=1}^{\infty} \underbrace{\frac{\langle f_n, h(x)\rangle}{\langle f_n, f_n\rangle}}_{b_n} f_n$$

Fourier coefficients are the coordinates of X!

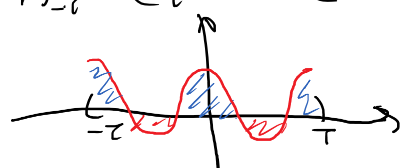
Base elements? $e_0 = 1$ $\qquad$ $e_n = \frac{1}{T}\cos\left(\frac{2\pi n}{T} t\right)$, $f_n = \frac{1}{T}\sin\frac{2\pi n}{T}t$

check with last week

$\langle e_n, e_m\rangle = 0$ if $n \neq m$ $\qquad 0 = \frac{1}{T^2}\int_{-T}^{T} \cos\left(\frac{2\pi n}{T} t\right)\cos\left(\frac{2\pi m}{T} t\right)$

$\langle e_n, f_n\rangle = 0 \qquad \langle e_n, e_m\rangle = 1$

$\langle f_n, f_n\rangle = 1$



use that $\frac{1}{2}(1 + \cos(2\alpha)) = \cos^2(\alpha)$ $\qquad \frac{1}{T^2}\int_{-T}^{T} \cos^2\left(\frac{2\pi n}{T} t\right) dt$

3.3. Fourier series for symmetric functions. Let f be a 2π-periodic function. Prove the following statements:

(a) If f is even, i.e. $f(-t) = f(t)\ \forall t$, then the real Fourier series of f has the following form

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kt).$$

(b) If f is odd, i.e. $f(-t) = -f(t)\ \forall t$, then the real Fourier series of f has the following form

$$\sum_{k=1}^{\infty} b_k \sin(kt).$$

3.3

f even. $\quad f(t) = f(-t)$

We need to calculte the "coordinates" aka. Fourier coefficients to write f as Fourier sum.

$$f(x) = \frac{\langle e_0, f\rangle}{\langle e_0, e_0\rangle} e_0 + \sum_n \overbrace{\frac{\langle e_n, f\rangle}{\langle e_n, e_n\rangle}}^{a_k} e_n + \sum_n \frac{\langle f_n, f\rangle}{\langle f_n, f_n\rangle} f_n$$

$f_n\, c_0$

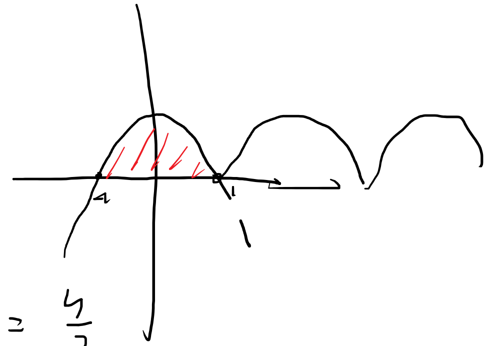
$$\int_{-\pi}^{\pi} \underbrace{f(t)}_{\text{even}} \underbrace{\overset{\sin}{\cos}\left(\frac{n\pi}{2\pi} t\right)}_{\text{odd.}} dt = 0 \quad \text{(proven before)}$$

$\to$ is odd! $\to$ s l

3.4. Fourier series I. Compute the real Fourier series (sine/cosine form) of the 2-periodic function

$$f(x) = 1 - x^2, \quad -1 < x < 1.$$

3.4 $\quad f(x) = 1 - x^2 \qquad -1 < x < 1$



→ Even, so $b_n = 0 \quad \forall n$.

$$a_0 = \int_{-1}^{1} f(x) \cdot 1 \, dx = \int_{-1}^{1} 1 - x^2 \, dx = \frac{4}{3}$$

$$a_n = \int_{-1}^{1} f(x) \frac{1}{2} \cos\left(\frac{2\pi n}{2} x\right) dx$$

$$\int_{-1}^{1} u v' = uv\Big|_{-1}^{1} - \int_{-1}^{1} u' v$$

$$= \int_{-1}^{1} (1 - x^2) \frac{1}{2} \cos\left(\frac{2\pi n}{2} x\right) dx$$

Integration by parts.

$$= \int_{-1}^{1} \frac{1}{2} \cos(\pi n x) - x^2 \cos(\pi n x) \, dx = \frac{4}{(n\pi)^2} (-1)^{n+1}$$