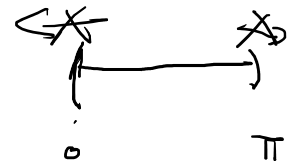

4.3. Heat equations with Neumann boundary conditions.

(a) Use a separation of variables Ansatz (i.e., writing u as a sum of solutions of the form $X(x)T(t)$) to solve the following PDE

$$\begin{cases} u_t - 4u_{xx} = 0 & (x,t) \in (0,\pi) \times \mathbb{R}_+ \\ u_x(0,t) = 0 & t \in \mathbb{R}_+ \\ u_x(\pi,t) = 0 & t \in \mathbb{R}_+ \\ u(x,0) = \sin(x) & x \in (0,\pi). \end{cases} \tag{1}$$

(b) The PDE (1) describes the heat propagation in an extremely thin channel. Calculate the following function

$$U(t) := \frac{1}{\pi} \int_0^\pi u(x,t)\,\mathrm{d}x.$$

One can interpret U as the average temperature of this channel. What can you deduce from U?

(c) Use a separation of variables Ansatz to solve the following PDE

$$\begin{cases} u_t - 4u_{xx} + u = 0 & (x,t) \in (0,\pi) \times \mathbb{R}_+ \\ u_x(0,t) = 0 & t \in \mathbb{R}_+ \\ u_x(\pi,t) = 0 & t \in \mathbb{R}_+ \\ u(x,0) = \sin(x) & x \in (0,\pi). \end{cases} \tag{2}$$

Compute the average temperature

$$\frac{1}{\pi} \int_0^\pi u(x,t)\,\mathrm{d}x$$

and the deviation of the temperature distribution

$$u(x,t) - \frac{1}{\pi} \int_0^\pi u(x,t)\,\mathrm{d}x.$$

Compare the behavior of the average temperature and of the deviation of the temperature distribution for this problem and the previous one.

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$u(x,t) = X(x)\,T(t)$

$X(x)\,T'(t) = 4\,X''(x)\,T(t)$ $\quad 0 < x < \pi,\ t > 0$

$X'(0)\,T(t) = 0$ $\quad t > 0$

$X'(\pi)\,T(t) = 0$ $\quad t > 0$

$$\frac{4X''(x)}{X(x)} = \frac{T'(t)}{T(t)}$$

only dep. on x — only dep on t.

$$\frac{4X''(x)}{X(x)} = \lambda = \frac{T'(t)}{T(t)}$$

constant

$\frac{d}{dt} \log(T(t))$

$$\rightarrow \quad X''(x) = \frac{\lambda}{4} X(x), \quad X'(0) = 0, \quad X'(\pi) = 0$$

2nd order linear ODE

$$X''(x) - \frac{\lambda}{4} X(x) = 0$$

a) $\lambda > 0$ $D > 0$ $\rightsquigarrow$ exponentials! $X(x) = A e^{\frac{1}{2}\sqrt{\lambda} x} + B e^{-\frac{1}{2}\sqrt{\lambda} x}$

b) $\lambda = 0$ $X''(x) = 0$ $\Rightarrow X$ is linear $\Rightarrow X(x) = A + Bx$

c) $\lambda < 0$ $D < 0$ $\rightsquigarrow$ cos & sin $\Rightarrow A\cos(\frac{1}{2}\sqrt{-\lambda}\,x) + B\sin(\frac{1}{2}\sqrt{-\lambda}\,x)$

$\hookrightarrow X'(x) = \frac{1}{2}\sqrt{-\lambda}\,(-A\sin(\frac{1}{2}\sqrt{-\lambda}\,x) + B\cos(\frac{1}{2}\sqrt{-\lambda}\,x))$ $|\lambda|$

a) ~~$u_x(x,0)$~~ $X'(0) = 0$ and $X'(\pi) = 0$ $\Rightarrow A = 0$, $B = 0$

$(e^{\frac{1}{2}\sqrt{\lambda} x}, e^{-\frac{1}{2}\sqrt{\lambda} x}$ lin ind.)

b) A arbitrary $B = 0$

c) $x = 0$: $\frac{1}{2} B \sqrt{-\lambda} = 0$ and $x = \pi$: $-\frac{1}{2} A\sqrt{-\lambda}\sin(\frac{1}{2}\sqrt{-\lambda}\,\pi) + \frac{1}{2} B\sqrt{-\lambda}\cos(\frac{1}{2}\sqrt{-\lambda}\,\pi) = 0$

only 0, if we have $\frac{1}{2}\sqrt{-\lambda}\,\pi = n\pi$.

0 of sin

$\lambda = -4n^2$ make dependent on n. $\frac{1}{2}\sqrt{-\lambda} = n$ $n \in \mathbb{Z}$.

depends on n

Eigenfunctions $A \cos(+\frac{1}{2}\sqrt{-\lambda}\,x)$

~~X_n~~ $C_n \cos(nx)$ $\rightsquigarrow$ still have to determine C_n

on the other hand

$$\frac{T'(t)}{T(t)} = -4n^2$$

$e^{\tilde{C}_n}$

$$\log(T_n(t)) = -4n^2 \, t + \tilde{C}_n$$

$$T_n(t) = \tilde{C}_n \, e^{-4n^2 t}$$

$$u(x,t) = \sum_{n=0}^{\infty} X_n(x) \, T_n(t) = \sum_{n=0}^{\infty} C_n \, e^{-4n^2 t} \cos(nx)$$

C_n — combined constants!

$$u(x,0) = \sum_{n=0}^{\infty} C_n \cos(nx) = \sin(x)$$

$e^{ins} + e^{-ins} = 2\cos(ns)$

Fourier transform! Take scalar product!

$C_n = \frac{1}{\pi}\int_0^{\pi} \underbrace{f(s)}_{\sin(s)} e^{-ins}\, ds$ $\qquad$ $C_{-n} = \frac{1}{\pi}\int_0^{\pi} \underbrace{f(s)}_{\sin(s)} e^{ins}\, ds$

$C_0 = \frac{1}{\pi}\int_0^{\pi} \underbrace{f(s)}_{\sin(s)}\, ds.$ $\qquad$ $C_n = \frac{1}{\pi}\int_0^{\pi} \sin(s)\, e^{-ins}\, ds$

i.b.p. $\int_0^{\pi} u v' = uv\Big|_0^{\pi} - \int_0^{\pi} u'v\, d$

$\hookrightarrow C_0 = \frac{2}{\pi}.$ $\qquad$ $\frac{2}{\pi}\frac{1+1}{1-4n^2}$ n even, n odd 0 $= \frac{1}{\pi}\frac{1+e^{-in\pi}}{1-4n^2}$ $\quad n \in \mathbb{Z}$

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$$\frac{1}{2}(c_n + c_{-n}) = c_n$$

$$u(x,t) = \frac{2}{\pi} - \sum_{n=1}^{\infty} \frac{4}{\pi} \frac{1}{1-4n^2} e^{-16n^2 t} \cos(2nx) \quad x \in [0,\pi]$$

Typo: should be $e^{-4n^2 t}$.

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One can interpret U as the average temperature of this channel. What can you deduce from U?

length of rod (not norm constant)

1/2

$\frac{2}{\pi^2}$

typo

$$\frac{1}{\pi}\int_0^\pi c_n\, e^{-4n^2 t} \cos(2nx)\,dx$$

we know these

$$\frac{\sin(2nx)}{2n}\Big|_0^\pi = 0$$

$c_0 = \frac{2}{\pi}$.

(c) Use a separation of variables Ansatz to solve the following PDE

$$\begin{cases} u_t - 4u_{xx} + u = 0 & (x,t) \in (0,\pi) \times \mathbb{R}_+ \\ u_x(0,t) = 0 & t \in \mathbb{R}_+ \\ u_x(\pi,t) = 0 & t \in \mathbb{R}_+ \\ u(x,0) = \sin(x) & x \in (0,\pi). \end{cases} \tag{2}$$

also by separation of variables

$$\frac{4X''(x)}{X(x)} = \frac{T'(t)}{T(t)} + 1 = \lambda$$

↳ seperable solve in the same way

Compute the average temperature

$$\frac{1}{\pi}\int_0^\pi u(x,t)\,\mathrm{d}x$$

$u_{new,avg}(t) = e^{-t} u_{old}$

and the deviation of the temperature distribution

$$u(x,t) - \frac{1}{\pi}\int_0^\pi u(x,t)\,\mathrm{d}x.$$

Temp − avg temp

$t > 0 \quad e^{-t} < 1 \Rightarrow u_{new}(x,t) = e^{-t} u_{old}(x,t)$

Compare the behavior of the average temperature and of the deviation of the temperature distribution for this problem and the previous one.

$Dev_{new} = e^{-t} Dev_{old}$ smaller deviations!

$$\frac{T_n'(t)}{T_n(t)} = -4n^2 - 1 \quad \hookrightarrow T_n(t) = e^{-4n^2 t - t}$$

↳ you get

$$\frac{2e^{-t}}{\pi} - \sum_{n=1}^{\infty} \frac{4}{\pi} \frac{1}{1-4n^2} e^{-(4n^2+1)t} \cos(2nx)$$

Typo