5.1. Nonhomogeneous to homogeneous. Please study Lesson 6 in Farlow's textbook. Consider the following nonhomogeneous general problem for a function $u : (0, L) \times (0, T) \to \mathbb{R}$

$$\begin{cases} u_t = u_{xx} & \text{in } (0, L) \times (0, T), \\ u(x, 0) = u_0(x) & \text{for } x \in [0, L], \\ \alpha_1 u(0, t) + \beta_1 u_x(0, t) = g_1(t) & \text{for } t \in (0, T), \\ \alpha_2 u(L, t) + \beta_2 u_x(L, t) = g_2(t) & \text{for } t \in (0, T), \end{cases}$$

linear

combination of $u(L,t) + \partial_x u(L,t)$

→ Heat eq

→ starting heat map

where $u_0 : [0, L] \to \mathbb{R}$, $g_1, g_2 : (0, T) \to \mathbb{R}$, $\alpha_{1,2}, \beta_{1,2} \in \mathbb{R}$ are given data.

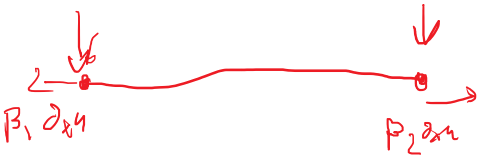
Under the technical assumption $L\alpha_1\alpha_2 + \alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$, discuss how to turn it into an equivalent one with homogeneous boundary conditions (the boundary conditions correspond to the last two equations).

Hint: Consider the problem satisfied by $U(x, t) := u(x, t) - A(t)(1 - \frac{x}{L}) - B(t)\frac{x}{L}$, where A, B are two functions (which you shall find) such that $A(t)(1 - \frac{x}{L}) + B(t)\frac{x}{L}$ satisfies the boundary conditions of the problem.

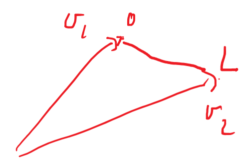
Inhomogeneous hard

↓ transform

Homogeneous: we know how to solve



$$S(x,t) = \underbrace{A(t)\left(1 - \frac{x}{L}\right) + B(t)\frac{x}{L}}_{\text{convex combination}}$$



$$\left(1 - \frac{x}{L}\right) v_1 + \frac{x}{L} v_2, \quad x \in [0, L]$$

$L = 1$

Last week small type $T_4(t) = e^{-4n^2 t}$

instead of $T_4(t) = e^{-16n^2 t}$.

Simpler case:

$\alpha_1 u(0,t) = k_1$ → constant

$\alpha_2 u(l,t) = k_2$

set to 1.

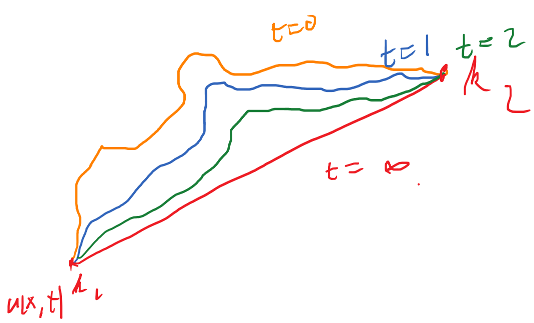
~~$u(x,t)$~~ ~~∂u~~

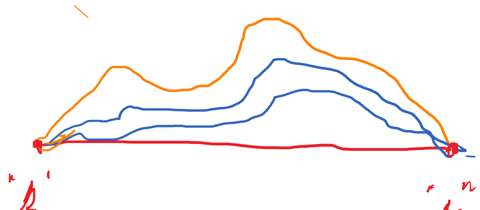
At $t=\infty$, we have

$u(0,\infty) = k_1$ $\quad u(L,\infty) = k_2$

$u(x,\infty) = k_1\left(1-\frac{x}{L}\right) + k_2\frac{x}{L} =$ "$S(x,t)$" → independent of t → steady state

$u(x,t) =$ "steady state" + "fluctuation" / "transient" → depends on the IV $u(x,0)$, starting heat map.





$u(x,t)$ − steady state $u(x,\infty)$

mapped back to hom. problem.

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$g_1(0)$ t

$g_1(t)$

$g_1(0)$

$g_0(x)$

Black magic: we still extropulate linearly but we introduce two time depent functions $A(t)$, $B(t)$ that hopefully will satisfy the equations for $g_1(t)$, $g_2(t)$ in the end. This gives

$$S(x,t) = A(t)\left(1 - \frac{x}{L}\right) + B(t)\frac{x}{L}$$

$U(x,t) = u(x,t) - S(x,t)$ will solve hom. problem

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$S(x,t) = \left(1 - \frac{x}{L}\right) A(t) + \frac{x}{L} B(t)$

$S(0,t) = A(t)$ $\quad \partial_x S(x,t) \overset{x=0,L}{=} \frac{B(t) - A(t)}{L}$

$S(L,t) = B(t)$

Any A, B s.t. S will solve BVP.

$\left(\alpha_1 - \frac{\beta_1}{L}\right) A(t) + \frac{\beta_1}{L} B(t) = g_1(t)$

$-\frac{\beta_2}{L} A(t) + \left(\alpha_2 + \frac{\beta_2}{L}\right) B(t) = g_2(t)$

$$\underbrace{\begin{pmatrix} \alpha_1 - \frac{\beta_1}{L} & \frac{\beta_1}{L} \\ -\frac{\beta_2}{L} & \alpha_2 + \frac{\beta_2}{L} \end{pmatrix}}_{M} \begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$$

det(M) has to be non zero in order to invert

$\det(M) = \left(\alpha_1 - \frac{\beta_1}{L}\right)\left(\alpha_2 + \frac{\beta_2}{L}\right) + \frac{\beta_1\beta_2}{L^2} = \frac{1}{L}\left(L\alpha_1\alpha_2 + \alpha_1\beta_2 - \alpha_2\beta_1\right)$

cancel

invert

$\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = M^{-1} \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$ these are given.

The transient solution $U(x,t) = u(x,t) - S(x,t)$ will satisfy

$\partial_t U = \partial_{xx}^2 U - \partial_t S$

$\partial_t(\overset{u(x,t)}{U + S})$

substitute $u(x,t) = U(x,t) + S(x,t)$ in the pde. $\left(\partial_x^2 S(x,t) = 0\right)$

$U(x,0) = U_0(x) - S(x,0)$

$\alpha_1 U(0,t) + \beta_1 \partial_x U(0,t) = 0$

$\alpha_2 U(L,t) + \beta_2 \partial_x U(L,t) = 0$

homogeneous problem

Exercise 2.

Thursday, October 28, 2021 9:22 AM

E.g. $u: [0,1] \times [0,\infty] \to \mathbb{R}$

$$\begin{cases} \partial_t u = \partial_{xx}^2 u & \text{in } (0,1) \times (0,\infty) \\ u(x,0) = x & x \in [0,1] \\ u(0,t) = 0 & t \in (0,\infty) \\ u(1,t) = \cos(t) & t \in (0,\infty) \end{cases}$$

plug in and use 5.1

$$\alpha_1 = \beta_1 = 0, \quad \alpha_2 = 1, \quad \beta_2 = 0$$

$$S(x,t) = + \frac{x}{1} \cos(t)$$

$(1) \to 1 \cdot 1$

$\cancel{t=0}$

$$U(x,t) = u(x,t) - \underbrace{x\cos(t)}_{S(x,t)}$$

$$\begin{cases} U(x,0) = 0 \\ U(0,t) = 0 \\ U(1,t) = 0 \\ \boxed{\partial_t U = \partial_{xx}^2 U + x\sin(t)} \end{cases}$$

$t=0$
$u(x,0) - x\overbrace{\cos(0)}^{=1} = 0$

Try to solve this with the techniques you know.