Thursday, November 4, 2021 8:46 AM

Fourier series on $[-T, T]$ $\xleftrightarrow{T\to\infty}$ $(-\infty, \infty)$ $x = \xi$

$$c_n = \frac{1}{2T} \int_{-T}^{T} f(s)\, e^{-\frac{i\pi n s}{T}}\, ds$$

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx}$$

$$\hat{f}(\xi) := \lim_{T\to\infty} (2T)\, c_n^{(T)} = \int_{-\infty}^{\infty} f(x)\, e^{-ix\xi}\, dx$$

$T \to \infty \quad \frac{\xi \pi n}{T} \in \frac{\pi}{T}\mathbb{Z} \approx \mathbb{R}$

$$f(x) = \frac{1}{2\pi} \frac{\pi}{T} \sum_{n\in\mathbb{Z}} 2T\, c_n^{(T)} e^{i\xi x}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{ix\xi}\, d\xi$$

In some books both integrals come with a norm. constant/prefactor $\frac{1}{\sqrt{2\pi}}$

(unitary vs. non-unitary)

Thursday, November 4, 2021 8:58 AM

$$F(f) = \int_{-\infty}^{\infty} f(x)\, e^{-ix\xi}\, dx$$

Key properties:

a) $F(f) = \hat{f}$ is linear. $F(cf + dg) = cF(f) + dF(g)$ $c, d \in \mathbb{R}$

i.b.p. n times

b) $\widehat{(x^n f)} = \left(-i \frac{d}{d\xi}\right)^n \hat{f}(\xi)$ $(-i)^n$

c) Affine substitutions: $\widehat{\left(f(a(x-d))\right)} = \frac{1}{a} e^{-i\xi d} \hat{f}\left(\frac{\xi}{a}\right)$

$a=1$ $\widehat{f}(x-d) = e^{-i\xi d} \hat{f}(\xi)$

$d=0$ $f(ax) = \frac{1}{a} \hat{f}\left(\frac{\xi}{a}\right)$.

d) Convolutions + products $f * g(x) = \int_{\mathbb{R}} f(y)\, g(x-y)\, dy$.

$\widehat{f*g}(x) = \hat{f} \cdot \hat{g}$ → products.

$\widehat{(f \cdot g)} = \frac{1}{2\pi} \hat{f} * \hat{g}$

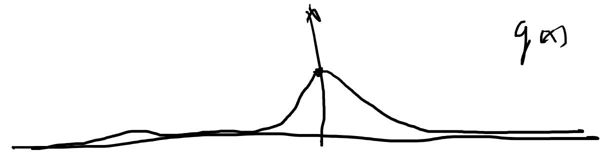
$a > 0$

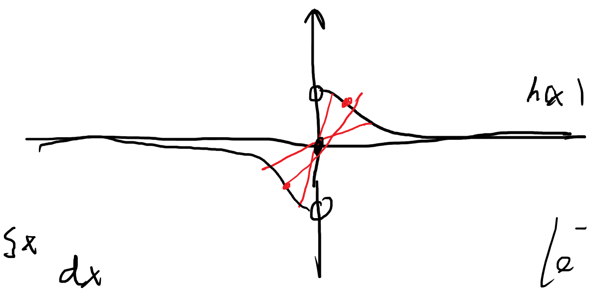
6.4. Computing Fourier transform on $\mathbb{R}$. Fix $a \neq 0$. Compute the Fourier transform of

$$g(x) = e^{-a|x|} \quad \text{and} \quad h(x) = \operatorname{sign}(x)e^{-a|x|},$$

where $\operatorname{sign}(x)$ is the sign function, that we here agree to be defined by

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$





$$\hat{g}(\xi) = \int_{\mathbb{R}} e^{-a|x| - i\xi x}\, dx$$

change of variables, $t \to -t$

$$= \int_{-\infty}^{0} e^{ax - i\xi x}\, dx + \int_{0}^{\infty} e^{-ax - i\xi x}\, dx$$

$$|e^{-i\xi x}| = 1$$

$$= \frac{e^{(a-i\xi)x}}{a - i\xi}\Bigg|_{-\infty}^{0} + \frac{e^{-ax - i\xi x}}{-a - i\xi}\Bigg|_{0}^{\infty}$$

$$0 \leq |e^{ax} e^{-i\xi x}| \leq e^{ax}$$

$$= \frac{1}{a - i\xi} - \lim_{R \to -\infty} \underbrace{\frac{e^{aR} e^{-i\xi R}}{a - i\xi}}_{=0} + \lim_{R \to \infty} \underbrace{\frac{e^{-aR - i\xi R}}{-a - i\xi}}_{=0} + \frac{1}{a + i\xi}$$

$$= \frac{1}{a - i\xi} - \frac{1}{a + i\xi} = \frac{2a}{a^2 + \xi^2}$$

Thursday, November 4, 2021 9:24 AM

$$\hat{h}(\xi) = \int_{\mathbb{R}} \operatorname{sign}(x) e^{-a|x| - i\xi x} dx = - \int_{-\infty}^{0} e^{ax - i\xi x} + \int_{0}^{\infty} e^{-ax - i\xi x}$$

!

same steps

$$= -\frac{2i\xi}{a^2+\xi^2}.$$

6.3 → compare with calculating values of sums by means of their associated Fourier series.

Thursday, November 4, 2021 9:27 AM

$f(x) = e^{-x^2}$

$$\widehat{e^{-x^2}}(\xi) = \int_{\mathbb{R}} e^{-x^2} e^{-ix\xi}\, dx$$

$$\int_{\mathbb{R}} x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

multipl. ⟷ derivatives

Show (ex 6.2)

$$\int_{\mathbb{R}} x^2 e^{-x^2} dx = -\frac{d^2}{d\xi^2} \widehat{e^{-x^2}}(0)$$

$$\boxed{\widehat{e^{-x^2}}(\xi) = \sqrt{\pi}\, e^{-\frac{\xi^2}{4}}}$$

$$\left(ix - \frac{\xi}{2}\right)^2 - \frac{\xi^2}{4} \qquad \int_{\mathbb{R}} e^{-(x^2 + ix\xi)} dx = \int_{\mathbb{R}} e^{\left(ix - \frac{\xi}{2}\right) - \frac{\xi^2}{4}} dx$$

$$a^2 + 2ab + b^2$$

$$(ix)^2 - 2ix\frac{\xi}{2} + \frac{\xi^2}{4}$$

completing the square

Put everything together

$$\int_{\mathbb{R}} x^2 e^{-x^2} dx = -\frac{d^2}{d\xi^2}\left(\sqrt{\pi}\, e^{-\frac{\xi^2}{4}}\right)\Big|_{\xi=0} = -\sqrt{\pi}\left(-\frac{1}{2} e^{-\frac{\xi^2}{4}}\right)\Big|_{\xi=0}$$

$$= \frac{\sqrt{\pi}}{2}$$

Evaluate Fourier series in a point (say $x=0$) ⇒ we get a specific sum

Evaluate Fourier transform in a point (say $\xi=0$) ⇒ we get a specific integral.