7.4. Solving an ODE with the Fourier transform. Find a solution $u : \mathbb{R} \to \mathbb{R}$ to the ODE

$$\boxed{-u''(x) + u(x)} = \boxed{e^{-|x|}}.$$

Hint: Take the Fourier transform of the whole ODE and recall that, for any integrable f, $\mathcal{F}(f * f) = \mathcal{F}(f)^2$.

$a = -1$, $b = 0$, $c = 1$, 2nd order, linear, inhomogeneous

(Notation: $\hat{u}(\xi) = F(u(x))(\xi)$)

Plan: 1.) Follow the hint: FT the entire equation.

Reminder: FT is linear.

$F(-u''(x) + u(x))(\xi) = -F(u''(x))(\xi) + F(u(x))(\xi)$

$F(u'(x)) = i\xi F(u(x))$

2.) $F(-u'') = \xi^2 \hat{u}$

$F(e^{-|x|}) = \dfrac{2}{1 + |\xi|^2}$ (see exercise 6.4)

3.) Perform the FT.

LHS: $F(-u'' + u) = (1 + \xi^2)\hat{u}$

RHS: $F(e^{-|x|}) = \dfrac{2}{1 + |\xi|^2}$

so $(1 + |\xi|^2)\hat{u} = \dfrac{2}{1 + |\xi|^2}$

Combined, we get: $\hat{u}(\xi) = \dfrac{2}{(1 + |\xi|^2)^2} = \dfrac{1}{2} \underbrace{\dfrac{4}{(1 + |\xi|^2)^2}}_{\substack{\text{recognize!} \\ (F(e^{-|x|}))^2}} \overset{\text{Conv. Property}}{=} \dfrac{1}{2} F(e^{-|x|} * e^{-|x|})$

4) Take inverse transform:

$u(x) = F^{-1}\left(\frac{1}{2} F(e^{-|x|} * e^{-|x|})\right) = \frac{1}{2} e^{-|x|} * e^{-|x|}$

cancel out

5) Calculate the convolution: split the integral

WLog $\boxed{x \geq 0}$ split over $[-\infty, 0]$, $[0, x]$, $[x, \infty)$

$\frac{1}{2} e^{-|x|} * e^{-|x|} = \frac{1}{2}\int_{\mathbb{R}} e^{-|y-x|} e^{-|y|}\, dy$ (definition of convolution)

$= \frac{1}{2}\int_{-\infty}^{0} e^{-|y-x|} e^{-|y|}\, dy + \frac{1}{2}\int e^{-|y-x|} e^{-|y|}\, dy + \frac{1}{2}\int e^{-|y-x|} e^{-|y|}\, dy$

Individual terms:

→ $\int_{-\infty}^{0} e^{-|y|} e^{-|y-x|}\, dy = e^{-x}\int_{-\infty}^{0} e^{y} e^{y}\, dy = e^{-x}\int_{-\infty}^{0} e^{2y}\, dy = e^{-x}\left[\frac{1}{2}e^{2y}\right]_{-\infty}^{0} = \frac{e^{-x}}{2}$

→ $\int_{0}^{x} e^{-|y|} e^{-|y-x|}\, dy = \int_{0}^{x} e^{-y} e^{y-x}\, dy = e^{-x}\int_{0}^{x} 1\, dy = x e^{-x}$

→ $\int_{x}^{\infty} e^{-|y|} e^{-|y-x|}\, dy = \int_{x}^{\infty} e^{-y} e^{x-y}\, dy = e^{x}\int_{x}^{\infty} e^{-2y}\, dy = \frac{e^{-x}}{2}$

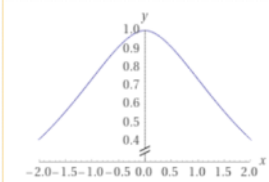
6) Putting it all back together.

$\frac{1}{2}(e^{-|x|} * e^{-|x|}) = \frac{1}{2}(1 + x)e^{-x}$ for $x \geq 0$

$= \frac{1}{2}\boxed{(1 + |x|)e^{-|x|}}$ for all x (do it again for $\boxed{x \leq 0}$).

7) Thus, $u(x) = \frac{1}{2}(1 + |x|)e^{-|x|}$

8) Question: is u C^2 ? Yes!



(show e.g. with a Taylor expans. at 0)

or simply take arbitrary derivates and show that they are continuous)

7.3. Fourier transform. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous integrable function and let $a \in \mathbb{R}$ be a real number. Define the function $f : \mathbb{R} \to \mathbb{R}$ as $f(x) := g(x+a) - g(x)$. Show that there are ~~infinitely~~ countably many values $\xi \in \mathbb{R}$ such that $\hat{f}(\xi) = 0$.

Reminder (again): (Convolution and modulation property) (i.e products become convolutions after FT, and vice versa)

see lecture 6 !

1.) $\mathcal{F}(f \cdot g) = \mathcal{F}(f) * \mathcal{F}(g)$ (conv. property)

2) $\mathcal{F}(f * g) = \frac{1}{2\pi} \mathcal{F}(f) \cdot \mathcal{F}(g)$ (modulation property)

Also: translations become modulations!

3.) $\mathcal{F}(f(c(x - a))) = \frac{1}{c} e^{-i\xi a} \hat{f}\left(\frac{\xi}{a}\right)$ ~> using this one

(changed notation to fit exercise)

$$\mathcal{F}(g(x+a) - g(x)) = e^{+i\xi a} \hat{g}(\xi) - \hat{g}(\xi) = \underbrace{(e^{i\xi a} - 1)}_{\text{if this is } 0} \hat{g}(\xi)$$

$\xi_k = \frac{2\pi k}{a}$ $k \in \mathbb{Z}$ $\Rightarrow e^{i\xi_k a} = 1$ Then $\mathcal{F}(f)(\xi) = 0$

Request ; 7.2.b)

7.2. Fourier transform. Compute the Fourier transform of the following functions

(a) $f(x) := x^2 e^{-2|x|}$

(b) $g(x) := \sin(2x+1)e^{-4(x+1)^2}$

Reminder : $\mathcal{F}(f(x-a))(\xi) = e^{-ia\xi}\mathcal{F}(f(x))(\xi)$ (i) (simple change of variables)

$\mathcal{F}(f(\lambda x))(\xi) = \frac{1}{|\lambda|}\mathcal{F}(f(\frac{x}{\lambda}))(\xi)$ (ii)

$\mathcal{F}(e^{iax}f(x))(\xi) = \mathcal{F}(f(x))(\xi - a)$ (iii)

Euler: $e^{ix} = \cos(x) + i\sin(x)$

$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$

$\sin(x-1) = \frac{e^{i(x-1)} - e^{-i(x-1)}}{2i} = \frac{e^{-i}e^{ix} - e^{i}e^{-ix}}{2i}$

Try and recast g in better form.

$\mathcal{F}(\sin(2x+1)e^{-4(x+1)^2}) = e^{i\xi}\mathcal{F}(\sin(2x-1)e^{-(2x)^2})$ (i)

$x \hookrightarrow x-1$ in the $\mathcal{F}$-integral

$= e^{i\xi}\left(\frac{1}{2}\mathcal{F}(\sin(x-1)e^{-x^2})(\frac{\xi}{2})\right)$ (ii)

$= \frac{e^{i\xi}}{4i}\left(e^{-i}\mathcal{F}(e^{ix}e^{-x^2})(\xi) - e^{i}\mathcal{F}(e^{-ix}e^{-x^2})\right)(\xi)$

$\mathcal{F}(e^{ix}e^{-x^2})(\xi) = \mathcal{F}(e^{-x^2})(\xi - 1) \wedge \mathcal{F}(e^{-ix}e^{-x^2})(\xi) = \mathcal{F}(e^{-x^2})(\xi+1)$

$\Rightarrow \mathcal{F}((g(x))(\xi) = \frac{e^{i\xi}}{4i}\left(e^{-i}\mathcal{F}(e^{-x^2})(\frac{\xi}{2}-1) - e^{i}\mathcal{F}(e^{-x^2})(\frac{\xi}{2}+1)\right)$

use $\mathcal{F}(e^{-x^2}) = \sqrt{\pi}e^{-\xi^2/4}$

$= \frac{\sqrt{\pi}e^{i\xi}}{4i}\left(e^{-i}e^{-(\xi/2-1)^2/4} - e^{i}e^{-(\xi/2+1)^2/4}\right)$