8.2. Separation of variables for the inhomogeneous wave equation. Solve the following PDE

$$\begin{cases} u_{tt} - u_{xx} = 1 & \text{for } x \in (0,\pi),\ t > 0, \\ u(0,t) = u(\pi,t) = 0 & \text{for } t > 0, \\ u_t(x,0) = 0 & \text{for } x \in (0,\pi), \\ u(x,0) = \sin(x) & \text{for } x \in (0,\pi). \end{cases}$$

Hint: We propose two different, but equivalent approaches to solve this problem (in fact, it may be excellent practice for you to solve the problem above in both ways and make sure you get a consistent outcome!).

First approach: exploit the superposition principle. You may follow these steps to solve the exercise:

1. Find a particular solution $v(x,t) = v(x)$ which does not depend on the time parameter t.
2. Let $w := u - v$ and check that it solves a *homogeneous* wave equation.
3. Employ the separation of variables method (as in **Exercise 8.1**) to find w.

1.) Particular solution: $v(x,t) = v(x)$ (t-indep)

$\underbrace{v_{tt}}_{=0} - v_{xx} = 1 \longrightarrow$ General solution:

$-v_{xx} = 1 \Rightarrow v(x) = -\frac{x^2}{2} + b x + c$

Determine b and c $\longrightarrow$ Boundary values: $v(0) = v(\pi) = 0$

$c = 0$ $\quad b = \pi/2$

2) Substract from general solution u gives homogeneous ODE.

$(u - v)_{tt} - (u - v)_{xx} = \underbrace{u_{tt} - u_{xx}}_{=1} - (\overbrace{v_{tt}}^{=0} - v_{xx})$

$= 1 + \underbrace{v_{xx}}_{-1} = 0$

Thus $w := u - v$ solves

$$\begin{cases} \partial_{tt} w - \partial_{xx} w = 0 \\ w(0,t) = w(\pi,t) = 0 \quad (t > 0) \\ \partial_t w(x,0) = 0 \quad x \in (0,\pi) \\ w(x,0) = \sin(x) - v(x) = \sin(x) + \frac{x^2}{2} - \frac{\pi}{2} x \end{cases}$$

3) Separation of variables

$w(t,x) = X(x)\, T(t) \rightsquigarrow \frac{X''(x)}{X(x)} = \boxed{\frac{T''(t)}{T(t)}} = k$

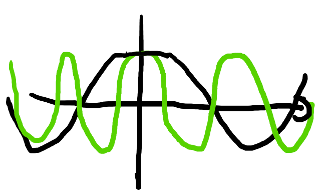
$\Rightarrow X''(x) = k\, X(x)$ and $X(0) = X(\pi) = 0$

↳ Eigenbasis: $\sin(nx)$ $\quad k = -n^2$

$\Rightarrow T''(t) = \overset{=-n^2}{k}\, T(t)$ and $\partial_t w(x,0) = 0$ $(T'(0) = 0)$

↳ Eigenbasis: $\cos(nt)$.

$\partial_t w(x,0) = T'(0)\, X(x) = 0$



$\frac{2}{\pi} \int_0^\pi \sin(nx) \sin(nx)\, dx = 1$

Superposition: $w(t,x) = \sum_{n=1}^{\infty} \boxed{A_n} \cos(nt) \sin(nx)$

How to determine the A_n?

We know: $w(x,t) = \sin(x) - v(x) = \sin(x) + \frac{x^2}{2} - \frac{\pi}{2} x$

$A_n = \frac{2}{\pi} \int_0^\pi \overbrace{(\sin(x) - v(x))}^{= w(x,t)} \sin(nx)\, dx$ $\quad a_1 = 1$, $a_n = 0$ $n > 1$

$\frac{2}{\pi} \int_0^\pi \sin(x) \sin(nx)\, dx = 0$

why?

only $\frac{2}{\pi} \int_0^\pi \sin(x) \sin(x)\, dx = 1$.

(trivial)

$b_n = \frac{2}{\pi} \int_0^\pi \left(\frac{x^2}{2} - \frac{\pi}{2} x \right) \sin(nx)\, dx$ $\quad n \geq 1$

$= \frac{2(-1 + (-1)^n)}{\pi n^3}$ $\quad \forall n \geq 1$

we get

$$w(x,t) = 1 \cdot \cos(t) \sin(x) + \sum_{n=1}^{\infty} \frac{2((-1) + (-1)^n)}{\pi n^3} \cos(nt) \cos(nx)$$

$$u(x,t) = v(x) + w(x,t) = -\frac{x^2}{2} + \frac{\pi}{2}x + \cos(t)\sin(x) + \sum_{n=1}^{\infty} \frac{2(-1+(-1)^n)}{\pi n^3} \cos(nt)\sin(nx)$$

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Hint: We propose two different, but equivalent approaches to solve this problem (in fact, it may be excellent practice for you to solve the problem above in both ways and make sure you get a consistent outcome!).

Second approach: perform an eigenfunction expansion, exactly as we had done in class for the inhomogeneous heat equation (cf. Exercise 2 in Lecture 5.)

In the case of the wave equation, you may benefit from reading Farlow's lesson 20, and then follow the very same strategy (i.e. the same steps).

Second approach:

$-\partial_{xx} u$, null boundary conditions $u(0,t) = 0 = u(\pi,t)$

$\rightsquigarrow \sin(nx)$ Eigenfunctions

Ansatz : $u(x,t) = \sum_{n\geq 1}^{\infty} a_n(t) \sin(nx)$

Inhom term $1.$ (const.)

$$b_n = \frac{2}{\pi}\int_0^{\pi} 1 . \sin(nx)\, dx = \begin{cases} \frac{2}{n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$1 = \frac{2}{\pi}\sum_{n \text{ odd}} \frac{2}{n} \sin(nx)$$

Now plug Ansatz into the ODE :

$$\sum_{n=1}^{\infty} a_n''(t) \sin(nx) + n^2 a_n(t) \sin(nx) = 1 = \sum_{n \text{ odd}} \frac{4}{\pi n} \sin(nx)$$

$$\sum_{n\geq 1}^{\infty} a_n'(0) \sin(nx) = \partial_t u(x,0) = 0$$

$$\sum_{n\geq 1} a_n(0) \sin(nx) = u(x,0) = \sin(x)$$

			Solutions
$n=1$	$a_1''(t) + a_1(t) = \frac{2}{\pi}$,	$a_1(0)$, $a_1'(0) = 0$	$a_1(t) = \cos(t) + \frac{4}{\pi}(1-\cos(t))$
n even	$a_n'' + n^2 a_n = 0$,	$a_n(0) = 0$, $a_n'(0) = 0$	$a_n(t) = 0$
n odd	$a_n'' + n^2 a_n = \frac{4}{\pi n}$,	$a_n(0) = 0$, $a_n'(0) = 0$	$a_n(t) = \frac{4}{n\pi^3}(1-\cos(nt))$

$$u(x,t) = \cos(t)\sin(x) + \frac{4}{\pi}\sum_{n\geq 1} \frac{1-\cos(nt)}{n^3} \sin(nx)$$

Check that they are the same

Are the solutions the same?

From 1st method:

$$v(x) = \frac{4}{\pi}\sum_{\substack{n\geq 1\\ \text{odd}}} \frac{\sin(nx)}{n^3}$$

expand in sin basis.

$\xrightarrow{\text{Plug}}$ back into

$$u(x,t) = \cos(t)\sin(x) + \sum_{n=1}^{\infty} \frac{2(-1+(-1)^n)}{\pi n^3} \cos(nt)\sin(nx) \boxed{+\ v(x)}$$

substitute