Live version

$-i\xi$

$\partial_{tt}\hat{u} + 4\xi^2 \hat{u}(\xi,t)$

$\hat{u}(t,\xi) = \int_{-\infty}^{\infty} u(x,t)\, e^{-ix\xi}\, dx$

$\partial_{tt} u - 4\, \partial_{xx} u = 0$

$c = 2$

$x \sin(4t)$

~~$v(x,t) = v_1(x)\, v_2(t)$~~

9.2. Inhomogeneous wave equation on the real line. Let $u : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ be a solution of the PDE

$$\begin{cases} \boxed{u_{tt} - 4u_{xx}} = \sin(4t) + x & \text{for } x \in \mathbb{R},\ t > 0, \\ u(x,0) = 2x^2 & \text{for } x \in \mathbb{R}, \\ u_t(x,0) = 6\cos(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

f $\quad g$

1. Find a particular solution v (which does not necessarily satisfy the initial conditions) by employing the Ansatz $v(x,t) = v_1(x) + v_2(t)$. why?
2. Write down the PDE satisfied by $w := u - v$.
3. Use d'Alembert's formula to find an explicit formula for w and deduce an explicit formula for u.

1) $\partial_{tt} v_2(t) - 4\, \partial_{xx} v_1(x) = \sin(4t) + x$

$$\begin{cases} -4\, v_1''(x) = x \\ v_2''(t) = \sin(4t) \end{cases}$$

easy! $\Rightarrow v_1(x) = -\dfrac{x^3}{24}$ $\qquad a x^3$

$\Rightarrow v_2(t) = -\dfrac{\sin(4t)}{16}$

$\partial_t (\cancel{v_1(x)} + v_2(t)) = v_2'(t) = -\dfrac{\cos(4t)}{4}$

$$v(x,t) = \underbrace{-\frac{x^3}{24}}_{v_1(x)} - \underbrace{\frac{\sin(4t)}{16}}_{v_2(t)}$$

2) $w = u - v$ $\rightsquigarrow$ transform into homogeneous problem.

$$\begin{cases} w_{tt} - 4 w_{xx} = 0 \\ 2x^2 - v_1(x) = 2x^2 + \dfrac{x^3}{24} \\ 6\cos(x) - v_2'(0) = 6\cos(x) + \dfrac{1}{4} \end{cases}$$

$(v_2(0) = 0)$

add in v in the IVP.

$f(x) = 2x^2 + \dfrac{x^3}{24}$

$g(x) = 6\cos(x) + \dfrac{1}{4}$

right traveling $\qquad$ left traveling

d'Alembert: $w(x,t) = \dfrac{1}{2}\Big[\underbrace{f(x-ct)}_{w(x-ct,0)} + \underbrace{f(x+ct)}_{w(x+ct,0)}\Big] + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} \underbrace{g(u)}_{6\cos(x) + \frac{1}{4}}\, du$

$c = 2$

$(a+b)^3 = a^3 + 3a^2 b + 3b^2 a + b^3$

$x^2 - \cancel{4tx} + 4t^2 \qquad x^2 + \cancel{4tx} + 4t^2$

$$= \frac{1}{2}\left(2(x-2t)^2 + \frac{(x-2t)^3}{24} + 2(x+2t)^2 + \frac{(x+2t)^3}{24}\right) + \frac{1}{4}\int_{x-ct}^{x+ct} 6\cos(u) + \frac{1}{4}\, du$$

$$= \frac{1}{2}\left(4x^2 + 16t^2 + \frac{(x-2t)^3}{24} + \frac{(x+2t)^3}{24}\right) + \frac{1}{4}\Big[6\sin(u) + \frac{1}{4}u\Big]_{u=x-2t}^{u=x+2t}$$

$$= \frac{1}{2}\left(4x^2 + 16t^2 + \frac{1}{24}(2x^3 + 24xt^2)\right) + \frac{1}{4}\Big(t + 6\underbrace{(\sin(x+2t) - \sin(x-2t))}\Big)$$

$\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha \sin\beta$

$$= 2x^2 + 8t^2 + \frac{x^3}{24} + \frac{1}{2}x t^2 + \frac{1}{4}t + 3\cos(x)\sin(2t)$$

$$u(x,t) = w(x,t) + v(x,t) = 2x^2 + 8t^2 + \cancel{\frac{x^3}{24}} + \frac{1}{2}xt^2 + \frac{1}{4}t + 3\cos(x)\sin(2t) - \cancel{\frac{x^3}{24}} - \frac{\sin(4t)}{16}$$

$|u(x,t)|^2$

$(8 + \frac{1}{2}x)\, t^2$

$x < -16 \qquad x \geq -16$

Re-recorded version.

Thursday, November 25, 2021 3:37 AM

$\hat{u}(\xi,t) = \int_{-\infty}^{\infty} u(x,t)\, e^{-ix\xi}\, dx$

$\partial_{tt}\hat{u} + \xi^2 \hat{u}$

$e^{i\xi(ct+x)}$

$u(x,t) = \int_{\mathbb{R}} \Big(\hat{f}(\xi) \underbrace{\cos(c\xi t)}_{\frac{e^{ic\xi t}+e^{-ic\xi t}}{2}} + \hat{g}(\xi) \underbrace{\sin(c\xi t)}_{\frac{e^{ic\xi t}-e^{-ic\xi t}}{2i}} \Big) e^{i\xi x}\, d\xi$

$\partial_{tt} u - 4\partial_{xx} u = 0$

$c = 2$

$\cancel{v_1(x)\, v_2(t)\ ? \Leftarrow x\sin(4t)}$

9.2. Inhomogeneous wave equation on the real line. Let $u : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ be a solution of the PDE

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$$\begin{cases} -4 v_1''(x) = x \\ v_2''(t) = \sin(4t) \end{cases}$$

easy! $\Rightarrow v_1''(x) = -\frac{x}{4} \Rightarrow v_1(x) = -\frac{x^3}{24}$

$\Rightarrow v_2(t) = -\frac{\sin(4t)}{16}$

$v(x,t) = \underbrace{-\frac{x^3}{24}}_{v_1(x)} \underbrace{- \frac{\sin(4t)}{16}}_{v_2(t)}$

2) $w = u - v \rightsquigarrow$ transform into homogeneous problem.

$$\begin{cases} w_{tt} - 4w_{xx} = 0 \\ f(u) \leftarrow w(x,0) = u(x,0) - v(x,0) = 2x^2 - v_1(x) - \overbrace{v_2(0)}^{0} = 2x^2 + \frac{x^3}{24} \\ g(u) \leftarrow w_t(x,0) = u_t(x,0) - v_t(x,0) = 6\cos(x) + \frac{\cos(4\cdot 0)}{4} = 6\cos(x) + \frac{1}{4} \end{cases}$$

d'Alembert: $w(x,t) = \frac{1}{2}\Big[\underbrace{f(x-ct)}_{\text{right traveling}} + \underbrace{f(x+ct)}_{\text{left traveling}} \Big] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(u)\, du$, $c=2$

$(a+b)^3 = a^3 + \cancel{3a^2b} + 3ab^2 + \cancel{b^3}$

$(a-b)^3 = a^3 - \cancel{3a^2b} + 3ab^2 - \cancel{b^3}$

$= \frac{1}{2}\Big(2\underbrace{(x-2t)^2}_{x^2 - 4tx + 4t^2} + \frac{(x-2t)^3}{24} + 2\underbrace{(x+2t)^2}_{x^2 + 4tx + 4t^2} + \frac{(x+2t)^3}{24} \Big) + \frac{1}{4}\int_{x-2t}^{x+2t} 6\cos(u) + \frac{1}{4}\, du$

$= \frac{1}{2}\Big(4x^2 + 16t^2 + \frac{1}{24}\big(2x^3 + \frac{6x(2t)^2}{24}\big)\Big) + \frac{1}{4}\Big[6\sin(u) + \frac{1}{4}u\Big]_{x-2t}^{x+2t}$

$= \Big(2x^2 + 8t^2 + \frac{x^3}{24} + \frac{1}{2}xt^2 \Big) + \frac{1}{4}\Big(6\underbrace{\big(\sin(x+2t) - \sin(x-2t)\big)}_{\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos(\alpha)\sin(\beta)} + t \Big)$

$= 2x^2 + 8t^2 + \frac{x^3}{24} + \frac{1}{2}xt^2 + 3\cos(x)\sin(2t) + \frac{1}{4}t$

Total solution!

$u(x,t) = w(x,t) + v(x,t) = 2x^2 + 8t^2 + \cancel{\frac{x^3}{24}} + \frac{1}{2}xt^2 + 3\cos(x)\sin(2t) + \frac{1}{4}t - \cancel{\frac{x^3}{24}} - \frac{\sin(4t)}{16}$

$|u(x,t)|^2$

$(8 - \frac{1}{2}x)t^2$

$x \le -16$

$\lim_{t\to\infty} u(x,t) = \begin{cases} +\infty & x \ge -16 \\ -\infty & x \le -16 \end{cases}$

$$\lim_{x \to -\infty} u(x, t) = +\infty$$