10.1. PDE under change of coordinates For each of the following 3 questions, you have to provide a *numeric or symbolic answer*. An example of a numeric answer is $-27\sqrt{2}$; an example of a symbolic answer is a^3/b or $u_t = u_{xx}$. Insert your answers in the following grid. Write clearly so that there is no ambiguity. You do **not** have to provide any justification for your answers.

Question	1	2	3
Answer			

Let $u(x,t)$ be a function satisfying $u_{xx} = u_{tt}$. For each of the following change of coordinates, write the PDE satisfied by $v(\xi,\eta) = u(\xi(x,t),\eta(x,t))$.

1. $\xi = 4x, \eta = 5t$.
2. $\xi = x + 3t, \eta = t$.
3. $\xi = 2x, \eta = xt$.

10.2. IVP under change of coordinates For each of the following 3 questions, you have to provide a *numeric or symbolic answer*. An example of a numeric answer is $-27\sqrt{2}$; an example of a symbolic answer is a^3/b or $u_t = u_{xx}$. Insert your answers in the following grid. Write clearly so that there is no ambiguity. You do **not** have to provide any justification for your answers.

Question	1	2	3
Answer			

Consider the following IVP:

$$\begin{cases} u_{tt} = u_{xx} & \text{for } x \in \mathbb{R} \text{ and } t > 0, \\ u(x,0) = f(x) & \text{for } x \in \mathbb{R}, \\ u_t(x,0) = g(x) & \text{for } x \in \mathbb{R},\ t > 0. \end{cases}$$

For each of the following change of coordinates, write the corresponding IVP for $v(\xi,\eta) = u(\xi(x,t),\eta(x,t))$.

1. $\xi = 4x, \eta = 5t$.
2. $\xi - x + 3t, \eta = t$.
3. $\xi = 2x, \eta = xt$.

10.3. Vanishing mixed derivative

Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a twice differentiable function such that $u_{xy} = 0$ vanishes identically. Then show that there exists (twice differentiable) functions $a, b : \mathbb{R} \to \mathbb{R}$ such that

$$u(x, y) = a(x) + b(y), \quad \text{for all } (x, y) \in \mathbb{R}^2.$$

(This is the reason why it is convenient, e.g. in studying the 1D wave equation, to introduce the *canonical coordinates*).

10.4. Pressure Wave

Consider the following equation:

$$\begin{cases} P_{tt} - P_{xx} = 0 & (x,t) \in \mathbb{R} \times \mathbb{R}_+ \\ P(x,0) = \begin{cases} 2, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases} \\ P_t(x,0) = \begin{cases} 1, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases} \end{cases}$$

(a) For which points in the domain does $P(x, t)$ automatically vanish? (Justify your answer with a drawing, without computing.)

(b) By using the d'Alembert's formula, compute $P(0, t)$ for all $t \geq 0$.

(c) *(Asymptotic behaviour for arbitrary x as $t \to \infty$)*. Show that there is a constant $P_\infty \neq 0$ such that

$$\lim_{t \to \infty} P(x, t) = P_\infty$$

for every x in $\mathbb{R}$. That means that the "pressure" in the whole room is constant and not equal to zero after the sound has subsided.

10.5. Initial value problem on the real line Consider the following PDE:

$$\begin{cases} u_{tt} - 2u_{tx} = 24u_{xx} & \text{for } x \in \mathbb{R},\ t > 0, \\ u(x,0) = 0 & \text{for } x \in \mathbb{R}, \\ u_t(x,0) = \sin(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

(a) Let $v : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be the function such that $v(x, t) = u(x - t, t)$ (or equivalently, consider the change of variables $(x', t') = (x - t, t)$). Compute the PDE satisfied by v.

(b) Using d'Alembert's formula, obtain a formula for v and deduce a formula for u.