12.1. Laplace equation For each of the following 5 statement you have to establish whether it is true or false.

Insert your answers in the following grid. Write clearly **T** if the statement is true and **F** if the statement is false. We will accept also **R** if the statement is *richtig* (which is the German word for *true*).

Only the answers in the grid will be taken into consideration for grading.

Question	1	2	3	4	5
Answer					

In this exercise, $\Delta = \partial_{xx} + \partial_{yy}$ is the Laplace operator in $\mathbb{R}^2$.

1. If $\Delta u = 0$ and $\Delta v = 0$ for all $(x, y) \in \mathbb{R}^2$, then for any real-valued smooth functions $c_1, c_2 : \mathbb{R}^2 \to \mathbb{R}$, we have
$$\Delta(c_1 u + c_2 v) = 0.$$
2. If $\Delta u = 0$ in $\mathbb{R}^2$, then $\Delta(\partial_x u) = 0$ in $\mathbb{R}^2$.
3. Let $D := \{(x, y) : x^2 + y^2 < 4\}$. There exist infinitely many functions $u : D \to \mathbb{R}$ such that
$$\begin{cases} \Delta u = 5 \quad \text{in } D, \\ u(1, 1) = 0. \end{cases}$$
4. Let $D := \{(x, y) : x^2 + y^2 = 1\}$. If $u : D \to \mathbb{R}$ solves
$$\begin{cases} \Delta u = 0 \quad \text{in } D, \\ u(x, y) = 2 - x^3 \quad \text{for } (x, y) \in \partial D, \end{cases}$$
then $u(0, \frac{1}{2}) > 1$.
5. Let $D := \{(x, y) : x^2 + y^2 = 1\}$. If $u : D \to \mathbb{R}$ solves
$$\begin{cases} \Delta u = 0 \quad \text{in } D, \\ u(x, y) = \sin(x) \quad \text{for } (x, y) \in \partial D, \end{cases}$$
then $u(0, 0) = \frac{1}{\pi}$.

12.2. Laplace equation in a disk Let $D := \{(x,y) : x^2+y^2<1\}$. We want to solve the Dirichlet boundary value problem

$$\begin{cases} \Delta u = 0 \quad \text{in } D, \\ u(x,y) = 1+3x^4 \quad \text{for } (x,y) \in \partial D. \end{cases}$$

(a) Prove $1 \le u(x,y) \le 4$ for all $(x,y) \in D$.

(b) Compute $u(0,0)$ with the Poisson formula.

(c) Compute u explicitly at all points $(x,y) \in D$ and prove $1 \le u(x,y) \le 4$ using this explicit expression.

12.3. Laplace equation in an annulus Let $D := \{(x,y) : 1 < x^2+y^2 < 4\}$. Find the solution $u : D \to \mathbb{R}$ of the following problem

$$\begin{cases} \Delta u = 0 \quad \text{in } D, \\ u(1,\theta) = 2\sin(2\theta) \quad \text{for } 0 \le \theta < 2\pi, \\ u(2,\theta) = 3\cos(3\theta) \quad \text{for } 0 \le \theta < 2\pi. \end{cases}$$