This problem set is meant to be an introduction to the use of polar coordinates.

2.1. Polar coordinates Every function $u : \mathbb{R}^2 \to \mathbb{R}$, $u = u(x, y)$ can be expressed in polar coordinates, i.e.

$$\begin{cases} x = x(r, \theta) = r\cos(\theta) \\ y = x(r, \theta) = r\sin(\theta) \end{cases}$$

then we write $u(r, \theta) = u(x(r, \theta), y(r, \theta))$.

(a) Rewrite the following function from Cartesian coordinates to polar coordinates

$$u_1(x, y) = x^2 + y^2, \qquad u_2(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \qquad u_3(x, y) = \frac{x}{y}.$$

(b) Rewrite the following function from polar coordinates to Cartesian coordinates

$$v_1(r, \theta) = r^n, \qquad v_2(r, \theta) = \sin\theta\cos\theta, \qquad v_3(r, \theta) = \theta \quad (-\pi/2 < \theta < \pi/2),$$
$$v_4(r, \theta) = \theta \quad (\pi/2 < \theta < 3\pi/2).$$

2.2. The Laplace operator in polar coordinates

Using the same notation to denote polar coordinates introduced in the previous exercise, show the following statements.

(a) Use chain rule to prove

$$\partial_r u(r, \theta) = (\partial_x u)\cos\theta + (\partial_y u)\sin\theta$$
$$\partial_\theta u(r, \theta) = -(\partial_x u) r\sin\theta + (\partial_y u) r\cos\theta.$$

Now we have the following relation for the partial derivatives $\partial_x u$ and $\partial_y u$:

$$\begin{pmatrix} \partial_r u \\ \partial_\theta u \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix}.$$

(b) By inverting some matrix, prove the following expressions for $\partial_x u$ and $\partial_y u$:

$$\partial_x u = \cos\theta(\partial_r u) - \frac{1}{r}\sin\theta(\partial_\theta u),$$
$$\partial_y u = \sin\theta(\partial_r u) + \frac{1}{r}\cos\theta(\partial_\theta u).$$

Use these formulas and chain rule to compute the direct expressions for $\partial^2_{xx} u$ and $\partial^2_{yy} u$ in polar coordinates, i.e.

$$\partial^2_{xx} u = \partial_x(\partial_x u) = \cos\theta(\partial_r(\partial_x u)) - \frac{1}{r}\sin\theta(\partial_\theta(\partial_x u)) = ...$$

(c) Combine all the information above and prove the following expression for the Laplacian operator in polar coordinates

$$\boxed{\Delta u(r, \theta) = \partial^2_{rr} u + \frac{1}{r} \partial_r u + \frac{1}{r^2} \partial^2_{\theta\theta} u.}$$

2.3. The heat equation on a thin disk Consider a thin (homogeneous) metal disk of radius $r_0 > 0$, whose temperature profile we shall describe in polar coordinates by means of a function $u(r, \theta, t)$. The initial temperature, at time t_0, is a known function $u_0 = u_0(r, \theta)$ and the body is completely insulated.

(a) Write down the full initial boundary value problem (IBVP) modelling the situation described above.

Tip: what is the exterior unit normal to a disk?

(b) What is the solution of this problem in the special case when the initial temperature is constant (equal to T_0)?

(c) Consider the problem in the special case when the initial temperature is a purely radial function, i.e. $u_0(r, \theta) = v_0(r)$. Make the ansatz that also the solution $u(r, \theta, t)$ does not depend on θ, i.e., $u(r, \theta, t) = v(r, t)$. Write down the equations satisfied by v. Prove that the quantity

$$\int_0^{r_0} r\, v(r, t)\, dr$$

does not depend on t. What is the phisical meaning of such quantity? Can you find the asymptotic state of this solution? The asymptotic state is the limit function $v_\infty(r) := \lim_{t\to\infty} v(r, t)$ and can be obtained by coupling the equations satisfied by v with the additional requirement $\partial_t v = 0$.