4.1. Fourier series II. Let $f : \mathbb{R} \to \mathbb{R}$ be the 2π-periodic even function such that

$$f(x) = e^x \quad \text{for} \quad x \in (0, \pi)\,.$$

Compute the complex Fourier series of f. Then, without any additional computation, determine the real Fourier series of the same function by employing the appropriate conversion formulae.

4.2. Convergent series. Let $f_e : \mathbb{R} \to \mathbb{R}$ be the 2-periodic *even* function such that $f_e(x) = x(1-x)$ for $x \in [0, 1]$. Let $f_o : \mathbb{R} \to \mathbb{R}$ be the 2-periodic *odd* function such that $f_o(x) = x(1-x)$ for $x \in [0, 1]$.

(a) Compute the Fourier series of f_e.

(b) Compute the Fourier series of f_o.

(c) Use the Fourier series of f_e to compute

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots\,.$$

(d) Use the Fourier series of f_o to compute

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \cdots\,.$$

4.3. Heat equations with Neumann boundary conditions.

(a) Use a separation of variables Ansatz (i.e., writing u as a sum of solutions of the form $X(x)T(t)$) to solve the following PDE

$$\begin{cases} u_t - 4u_{xx} = 0 & (x,t) \in (0,\pi) \times \mathbb{R}_+ \\ u_x(0,t) = 0 & t \in \mathbb{R}_+ \\ u_x(\pi,t) = 0 & t \in \mathbb{R}_+ \\ u(x,0) = \sin(x) & x \in (0,\pi). \end{cases} \tag{1}$$

(b) The PDE (1) describes the heat propagation in an extremely thin channel. Calculate the following function

$$U(t) := \frac{1}{\pi} \int_0^\pi u(x,t)\,\mathrm{d}x.$$

One can interpret U as the average temperature of this channel. What can you deduce from U?

(c) Use a separation of variables Ansatz to solve the following PDE

$$\begin{cases} u_t - 4u_{xx} + u = 0 & (x,t) \in (0,\pi) \times \mathbb{R}_+ \\ u_x(0,t) = 0 & t \in \mathbb{R}_+ \\ u_x(\pi,t) = 0 & t \in \mathbb{R}_+ \\ u(x,0) = \sin(x) & x \in (0,\pi). \end{cases} \tag{2}$$

Compute the average temperature

$$\frac{1}{\pi}\int_0^\pi u(x,t)\,\mathrm{d}x$$

and the deviation of the temperature distribution

$$u(x,t) - \frac{1}{\pi}\int_0^\pi u(x,t)\,\mathrm{d}x.$$

Compare the behavior of the average temperature and of the deviation of the temperature distribution for this problem and the previous one.