5.1. Nonhomogeneous to homogeneous. Please study Lesson 6 in Farlow's textbook. Consider the following nonhomogeneous general problem for a function $u : (0, L) \times (0, T) \to \mathbb{R}$

$$\begin{cases} u_t = u_{xx} & \text{in } (0,L) \times (0,T), \\ u(x,0) = u_0(x) & \text{for } x \in [0,L], \\ \alpha_1 u(0,t) + \beta_1 u_x(0,t) = g_1(t) & \text{for } t \in (0,T), \\ \alpha_2 u(L,t) + \beta_2 u_x(L,t) = g_2(t) & \text{for } t \in (0,T), \end{cases}$$

where $u_0 : [0, L] \to \mathbb{R}$, $g_1, g_2 : (0, T) \to \mathbb{R}$, $\alpha_{1,2}, \beta_{1,2} \in \mathbb{R}$ are given data.

Under the technical assumption $L\alpha_1\alpha_2 + \alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$, discuss how to turn it into an equivalent one with homogeneous boundary conditions (the boundary conditions correspond to the last two equations).

Hint: Consider the problem satisfied by $U(x, t) := u(x, t) - A(t)(1 - \frac{x}{L}) - B(t)\frac{x}{L}$, where A, B are two functions (which you shall find) such that $A(t)(1 - \frac{x}{L}) + B(t)\frac{x}{L}$ satisfies the boundary conditions of the problem.

5.2. Nonhomogeneous heat equation Solve the following problem for $u : [0, 1] \times [0, \infty] \to \mathbb{R}$

$$\begin{cases} u_t = u_{xx} & \text{in } (0,1) \times (0,\infty), \\ u(x,0) = x & \text{for } x \in [0,1], \\ u(0,t) = 0 & \text{for } t \in (0,\infty), \\ u(1,t) = \cos(t) & \text{for } t \in (0,\infty). \end{cases}$$

Hint: First transform the problem into one with null boundary conditions, then solve it by expanding it in terms of eigenfunctions.

5.3. Another variation on the heat equation Solve the following problem for $u : [0, 1] \times [0, \infty] \to \mathbb{R}$

$$\begin{cases} u_t = u_{xx} + \sin(\lambda_1 x) & \text{in } (0,1) \times (0,\infty), \\ u(x,0) = 0 & \text{for } x \in [0,1], \\ u(0,t) = 0 & \text{for } t \in (0,\infty), \\ u(1,t) + u_x(1,t) = 0 & \text{for } t \in (0,\infty), \end{cases}$$

where $\lambda_1 > 0$ is the smallest positive solution of $\tan(\lambda) = -\lambda$.

Hint: Use the method of expanding the solution in terms of eigenfunctions.