6.1. Integral given the Fourier transform Let f be an integrable function such that

$$\hat{f}(\xi) = \frac{1}{3 + 4\xi^2}\,.$$

Compute

$$\int_{\mathbb{R}} f(x)\,dx\,.$$

6.2. Moments of function given the Fourier transform Let f be an integrable function such that

$$\hat{f}(\xi) = \frac{3}{5 + i\xi}.$$

Compute the following integrals:

$$\int_{\mathbb{R}} f(x)\,\mathrm{d}x, \quad \int_{\mathbb{R}} x\,f(x)\,\mathrm{d}x, \quad \int_{\mathbb{R}} x^2\,f(x)\,\mathrm{d}x.$$

6.3. Tricky integral via Fourier transform With the help of the Fourier transform of $f(x) = \mathrm{e}^{-x^2}$, that has been computed in class (cf. Lecture 6), prove that

$$\int_{\mathbb{R}} x^2 \mathrm{e}^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}.$$

6.4. Computing Fourier transform on $\mathbb{R}$. Fix $a \neq 0$. Compute the Fourier transform of

$$g(x) = \mathrm{e}^{-a|x|} \quad \text{and} \quad h(x) = \operatorname{sign}(x)\mathrm{e}^{-a|x|},$$

where $\operatorname{sign}(x)$ is the sign function, that we here agree to be defined by

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$