7.1. Fourier transform of the indicator function of an interval As in Lecture 7, let $f : \mathbb{R} \to \mathbb{R}$ be the indicator function of $(-1, 1)$, namely

$$f(x) := \begin{cases} 1 & \text{if } -1 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute the convolution product $f * f$.

(b) Without performing any explicit computation (you may use only part **(a)** and the basic properties of the Fourier transform) determine the Fourier transform of the function

$$g(x) := \begin{cases} T + C - x & \text{if } C \leq x \leq C + T, \\ T - C + x & \text{if } C - T \leq x \leq C, \\ 0 & \text{otherwise,} \end{cases}$$

in terms of the positive parameters $T, C > 0$.

7.2. Fourier transform. Compute the Fourier transform of the following functions

(a) $f(x) := x^2 e^{-2|x|}$

(b) $g(x) := \sin(2x + 1)e^{-4(x+1)^2}$

Hint: For **(a)**, compute the Fourier transform of $h(x) = e^{-2|x|}$ and find the connection with the Fourier transform of f. For **(b)** use the basic properties of the Fourier transform to reduce the problem to the computation of the Fourier transform of e^{-x^2} (which was performed in class).

7.3. Fourier transform. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous integrable function and let $a \in \mathbb{R}$ be a real number. Define the function $f : \mathbb{R} \to \mathbb{R}$ as $f(x) := g(x + a) - g(x)$. Show that there are infinitely many values $\xi \in \mathbb{R}$ such that $\hat{f}(\xi) = 0$.

7.4. Solving an ODE with the Fourier transform. Find a solution $u : \mathbb{R} \to \mathbb{R}$ to the ODE

$$-u''(x) + u(x) = e^{-|x|}.$$

Hint: Take the Fourier transform of the whole ODE and recall that, for any integrable f, $\mathcal{F}(f * f) = \mathcal{F}(f)^2$.