8.1. Separation of variables for the homogeneous wave equation. Solve the following PDE

$$\begin{cases} u_{tt} - 4u_{xx} = 0 & \text{for } x \in (0,\pi),\ t > 0, \\ u(0,t) = u(\pi,t) = 0 & \text{for } t > 0, \\ u_t(x,0) = 0 & \text{for } x \in (0,\pi), \\ u(x,0) = f(x) & \text{for } x \in (0,\pi), \end{cases}$$

where the function $f : [0,\pi] \to \mathbb{R}$ is defined as follows:

$$f(x) = \begin{cases} x & \text{for } \ 0 \le x < \dfrac{\pi}{2}\,, \\ \pi - x & \text{for } \ \dfrac{\pi}{2} \le x \le \pi\,. \end{cases}$$

Hint: Proceeding exactly as we did for the heat equation (cf. Lecture 3), you will see that imposing the initial conditions requires you to expand the function $f(x)$ given above only in terms of $\sin(\cdot)$ functions (i.e. employing the basis $\{\sin(nx),\ n = 1,2,\ldots\}$); that is to say (equivalently): one needs compute the Fourier series *of the odd extension of f with period 2π.*

8.2. Separation of variables for the inhomogeneous wave equation. Solve the following PDE

$$\begin{cases} u_{tt} - u_{xx} = 1 & \text{for } x \in (0,\pi),\ t > 0, \\ u(0,t) = u(\pi,t) = 0 & \text{for } t > 0, \\ u_t(x,0) = 0 & \text{for } x \in (0,\pi), \\ u(x,0) = \sin(x) & \text{for } x \in (0,\pi). \end{cases}$$

Hint: We propose two different, but equivalent approaches to solve this problem (in fact, it may be excellent practice for you to solve the problem above in both ways and make sure you get a consistent outcome!).

First approach: exploit the superposition principle. You may follow these steps to solve the exercise:

1. Find a particular solution $v(x,t) = v(x)$ which does not depend on the time parameter t.
2. Let $w := u - v$ and check that it solves a *homogeneous* wave equation.
3. Employ the separation of variables method (as in **Exercise 8.1**) to find w.

Second approach: perform an eigenfunction expansion, exactly as we had done in class for the inhomogeneous heat equation (cf. Exercise 2 in Lecture 5.)

In the case of the wave equation, you may benefit from reading Farlow's lesson 20, and then follow the very same strategy (i.e. the same steps).