1.1. Linear ODE with constant coefficients. Solve the following differential equations for $y(x)$:

(a) $y'' - \omega^2 y = 0$,

(b) $y'' + \omega^2 y = 0$,

(c) $y'' + 3y' + 4y = \cos(2x)$.

Solution:

(a) If we insert $y(x) = e^{\lambda x}$ into the differential equation, we obtain the characteristic polynomial $\chi(\lambda) = \lambda^2 - \omega^2$, which has two zeros $\lambda_1 = \omega$ and $\lambda_2 = -\omega$.

Thus the general solution of this differential equation is

$$y(x) = Ae^{\omega x} + Be^{-\omega x},$$

with constants A, $B \in \mathbb{R}$. The solution can also be written as

$$y(x) = C\sinh(\omega x) + D\cosh(\omega x), \quad C, D \in \mathbb{R}.$$

(b) If we insert $y(x) = e^{\lambda x}$ into the differential equation, we obtain the characteristic polynomial $\chi(\lambda) = \lambda^2 + \omega^2$, with the two zeros $\lambda_1 = i\omega$ and $\lambda_2 = -i\omega$. Thus the general real solution of this differential equation is given by

$$y(x) = A\sin(\omega x) + B\cos(\omega x), \quad A, \ B \in \mathbb{R}.$$

(c) The characteristic polynomial of the homogeneous problem is $\chi(\lambda) = \lambda^2 + 3\lambda + 4$ and has the zeros $\lambda_{1,2} = \frac{1}{2}\left(-3 \pm i\sqrt{7}\right)$. Thus the general solution of the homogeneous problem is

$$y_h(x) = Ae^{-\frac{3}{2}x}\sin\left(\frac{\sqrt{7}}{2}x\right) + Be^{-\frac{3}{2}x}\cos\left(\frac{\sqrt{7}}{2}x\right).$$

We compute the special solution using the Ansatz

$$y_p(x) = A\cos(2x) + B\sin(2x).$$

Then we have

$$-6A\sin(2x) + 6B\cos(2x) = \cos(2x).$$

Thus, the special solution is $y_p(x) = \frac{1}{6}\sin(2x)$ and thus the general solution is

$$y(x) = Ae^{-\frac{3}{2}x}\sin\left(\frac{\sqrt{7}}{2}x\right) + Be^{-\frac{3}{2}x}\cos\left(\frac{\sqrt{7}}{2}x\right) + \frac{1}{6}\sin(2x).$$

1.2. First-order ODE with variable coefficients. Solve the following differential equations for $y(x)$:

(a) $y' - x^2 y = 0$, $x \in \mathbb{R}$,

(b) $y' - y/x = x$, $x > 0$,

(c) $y' + x^5 y = x^6 + 1$, $x \in \mathbb{R}$,

(d) $y' = (x + y)^2$,

(e) $y' - y = \sin x$,

(f) $y\, y' - (1 + y)x^2 = 0$.

Tips: ODE of 1st order may be solved by *separation of variables* or by substitution. For (c), multiply the equation with $\mathrm{e}^{f(x)}$, where f is a suitable function. For (f), y is not an explicit function of x. It is enough to write a relation between the function y and the variable x that does not contain any derivatives of y.

Solution:

(a) Assuming that the value of the solution at a point is $y \neq 0$, we have

$$\frac{y'}{y} = x^2 \implies \frac{\mathrm{d}\log(|y|)}{\mathrm{d}x} = x^2 \implies \log(|y|) = C + \frac{x^3}{3} \implies |y(x)| = e^{\frac{1}{3}x^3 + C} = Ke^{\frac{1}{3}x^3}\,.$$

We notice that the constant solution $y(x) = 0$ is also valid. Therefore the general solution is given by

$$y(x) = Ke^{\frac{1}{3}x^3}\,, \quad K \in \mathbb{R}\,.$$

(b) First we search for a solution of the homogeneous problem $y' - y/x = 0$. Exactly with the same reasoning used in **(a)**, we get

$$\log(|y|) = C + \int \frac{\mathrm{d}x}{x} \implies |y(x)| = e^C x = Kx\,,$$

hence, the solution of the homogeneous equation is $y = (\pm)x = Kx$ with $K \in \mathbb{R}$.

The solution of the inhomogeneous equation can be found with the ansatz $y(x) = K(x)x$. By inserting this into the equation we obtain $K'(x)x = x$. Thus $K(x) = x$ provides a special solution of the inhomogeneous problem and therefore the general solution of the inhomogeneous equation is

$$y(x) = x^2 + Kx.$$

(c) Multiplying the equation with the strictly positive function $e^{\frac{x^6}{6}}$ gives

$$e^{\frac{x^6}{6}}y'(x) + x^5 e^{\frac{x^6}{6}}y(x) = e^{\frac{x^6}{6}}(x^6+1).$$

Notice that the left side can be written as $\left(e^{\frac{x^6}{6}}y\right)'$. For the term on the right side, note that

$$e^{\frac{x^6}{6}}(x^6+1) = x\cdot(x^5e^{\frac{x^6}{6}}) + 1\cdot e^{\frac{x^6}{6}} = (xe^{\frac{x^6}{6}})',$$

Hence it follows that $e^{\frac{x^6}{6}}y(x) - xe^{\frac{x^6}{6}}$ is a constant function. The general solution of the differential equation is

$$y(x) = Ce^{-\frac{x^6}{6}} + x,$$

for any $C \in \mathbb{R}$.

(d) We use the substitution $z = x+y$, i.e. $z' = 1+y'$. The ODE then becomes $z' = z^2+1$, i.e. separable. By separating the variables we get:

$$\frac{\mathrm{d}z}{\mathrm{d}x} = z^2+1 \Rightarrow \int \frac{\mathrm{d}z}{z^2+1} = \int \mathrm{d}x \Rightarrow \arctan z = x + c, \quad C \in \mathbb{R}.$$

Thus we obtain the solution: $z(x) = \tan(x+C),\ C\in\mathbb{R}$, so $y(x) = \tan(x+C) - x$, $C \in \mathbb{R}$.

(e) We multiply both sides by $\rho(x) = \mathrm{e}^{-x}$ to get

$$\begin{aligned}\mathrm{e}^{-x}y'(x) - \mathrm{e}^{-x}y = \mathrm{e}^{-x}\sin x &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left(y(x)\mathrm{e}^{-x}\right) = \mathrm{e}^{-x}\sin x\\ &\Rightarrow y(x) = \mathrm{e}^{x}\int \mathrm{e}^{-x}\sin x\,\mathrm{d}x + C\mathrm{e}^{x} \quad C\in\mathbb{R}.\end{aligned}$$

The indefinite integral is calculated by partial integration:

$$\int \mathrm{e}^{-x}\sin x\,\mathrm{d}x = -\frac{1}{2}\mathrm{e}^{-x}(\sin x + \cos x),$$

thus we get the solution: $y(x) = -\frac{1}{2}(\sin x+\cos x) + C\mathrm{e}^{x}$, $C\in\mathbb{R}$.

(f) We see that the constant function $y \equiv -1$ solves the ODE. Now we look for the non-constant solutions. By separating the variables we get:

$$y\,\frac{\mathrm{d}y}{\mathrm{d}x} = (1+y)x^2 \Rightarrow \int \frac{y}{1+y}\,\mathrm{d}y \quad = \int x^2\,\mathrm{d}x \Rightarrow \int\left(1 - \frac{1}{1+y}\right)\mathrm{d}y = \int x^2\,\mathrm{d}x,$$

thus we get the implicit relation for the non-constant solutions of the ODE:

$$y - \log|1+y| = \frac{x^3}{3} + C, \quad C\in\mathbb{R}.$$

1.3. Initial and boundary value problems. Solve the following problems:

(a) $$\begin{cases} y' &= 2\mathrm{e}^{2x} \quad \forall x \in \mathbb{R}, \\ y(0) &= 2. \end{cases}$$

(b) $$\begin{cases} y''(x) + 4y(x) &= 0 \quad \forall x \in (0, L) \ (L > 0 \text{ given}), \\ y(0) &= 0, \\ y(L) &= 2. \end{cases}$$

Solution:

(a) We can get the general solution of ODE by direct integration:

$$y' = 2\mathrm{e}^{2x} \Rightarrow y(x) = \int 2\mathrm{e}^{2x}\,\mathrm{d}x + C \Rightarrow y(x) = \mathrm{e}^{2x} + C, \quad C \in \mathbb{R}.$$

The initial condition requires $2 = y(0) = 1 + C$, so $C = 1$ and the solution of the initial value problem is

$$y(x) = \mathrm{e}^{2x} + 1.$$

(b) The general solution of the differential equation $y'' + 4y = 0$ has the following form:

$$y(x) = A\sin(2x) + B\cos(2x).$$

The boundary conditions are

$$0 = y(0) = B,$$
$$2 = y(L) = A\sin(2L) + B\cos(2L).$$

Since $B = 0$, the second equation is $A\sin(2L) = 2$. This equation has exactly then a solution for A, when $\sin(2L) \neq 0$. So we have, for $L = \frac{\pi}{2}n$ with $n \in \mathbb{Z}$ the problem has no solution. For all other $L \in \mathbb{R}$ the unique solution of the boundary value problem is given by

$$y(x) = \frac{2}{\sin(2L)}\sin(2x).$$

1.4. Spring pendulum A spring pendulum consists of a coil spring and a mass test piece (with mass m) attached to it, which can move in a straight line in the direction in which the spring extends or retracts. Let $K > 0$ be the spring constant and $\omega^2 := K/m$, then the equation of motion of the spring pendulum is given by

$$\ddot{x}(t) + \omega^2 x(t) = 0. \tag{1}$$

Find the solution of the differential equation (1):

(a) with the initial conditions $x(0) = 1, \dot{x}(0) = 2\omega$.

(b) with the boundary conditions $x(0) = 1, x(\frac{\pi}{2\omega}) = 1$.

Solution: The zeros of the characteristic polynomial are

$$p(\lambda) = \lambda^2 + \omega^2 = 0 \Rightarrow \lambda_{1,2} = \pm i\omega.$$

Thus the general solution is given by

$$x_{all}(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t), \qquad C_1, C_2 \in \mathbb{R}.$$

(a) With the initial conditions $x(0) = 1$ and $\dot{x}(0) = 2\omega$ we have

$$\begin{aligned} x(0) = 1 \quad &\Rightarrow \quad C_1 \cos(0) + C_2 \sin(0) = C_1 = 1, \\ \dot{x}(0) = 2\omega \quad &\Rightarrow \quad -C_1\omega \sin(0) + C_2\omega \cos(0) = C_2\omega = 2\omega. \end{aligned}$$

Thus $C_1 = 1$ and $C_2 = 2$. Then the solution is

$$x(t) = \cos(\omega t) + 2\sin(\omega t).$$

(b) With the boundary conditions $x(0) = 1$ and $x(\frac{\pi}{2\omega}) = 1$ we have

$$\begin{aligned} x(0) = 1 \quad &\Rightarrow \quad C_1 \cos(0) + C_2 \sin(0) = C_1 = 1, \\ x(\frac{\pi}{2\omega}) = 1 \quad &\Rightarrow \quad C_1 \cos(\frac{\pi}{2}) + C_2 \sin(\frac{\pi}{2}) = C_2 = 1. \end{aligned}$$

Thus $C_1 = C_2 = 1$. Then the solution is

$$x(t) = \cos(\omega t) + \sin(\omega t).$$

1.5. Classification of PDEs I. Suppose a, b, f and g are differentiable functions. Tell whether the following differential equations in $u(x, y)$ are linear and homogeneous, linear and inhomogeneous, or non-linear and tell their order. For every linear differential equation of 2nd order, tell whether the equation is elliptic, hyperbolic or parabolic.

(a) $u_{xxx} + u_y = f$

(b) $au_{xx} + bu^2 = 0$

(c) $u_x u_y = 0$

(d) $2u_{xx} + u_x + 2u_{xy} + 2u_{yy} = 0$

(e) $(1 - x^2)u_{xx} - 2xyu_{xy} + (1 - y^2)u_{yy} = g$ in $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 > 1\}$.

Solution:

- **(a)** linear and inhomogeneous, 3rd order
- **(b)** non-linear, 2nd order
- **(c)** non-linear, 1st order
- **(d)** linear and homogeneous, 2nd order. The discriminant of the equation is (in this case $a = 2$, $b = 2$, $c = 2$) $b^2 - 4ac = -12 < 0$, so the PDE is elliptic.
- **(e)** linear and inhomogeneous, 2nd order. The discriminant of the equation is (in this case $a = 1 - x^2$, $b = 2xy$, $c = 1 - y^2$)

$$b^2 - 4ac = 4(x^2y^2 - (1 - x^2)(1 - y^2)) = 4(x^2 + y^2 - 1) > 0 \quad \text{in}\, \Omega,$$

so the PDE is hyperbolic.

1.6. Classification of PDEs II.

Suppose a, b and g differentiable functions with $g > 0$.

Tell whether the following differential equations in $u(x, y)$ are linear and homogeneous, linear and inhomogeneous, or non-linear and tell their order. For every linear differential equation of 2nd order, tell whether the equation is elliptic, hyperbolic or parabolic.

- **(a)** $au_{xxx} + b(u^4 + u) = 0$
- **(b)** $a^2u_{xx} + u_xu_y = 1$
- **(c)** $4u_{xx} + u_x + u_{xy} + 6u_{yy} = 0$
- **(d)** $(x^2 - 2)u_{xx} + 4xyu_{xy} + (y^2 - 2)u_{yy} = g$ in $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 > 16\}$.

Solution:

- **(a)** non-linear, 3rd order
- **(b)** non-linear, 2nd order
- **(c)** linear, homogeneous, 2nd order. The discriminant of the equation is (in this case $a = 4, b = 1$, $c = 6$) $b^2 - 4ac = -95 < 0$, so the PDE is elliptic.
- **(d)** linear, inhomogeneous, 2nd order. The discriminant of the equation is (in this case $a = x^2 - 2$, $b = 4xy$, $c = y^2 - 2$)

$$\begin{aligned} b^2 - 4ac &= 4\big(4x^2y^2 - (x^2 - 2)(y^2 - 2)\big) = 4\big(2(x^2 + y^2) - 4 + 3x^2y^2\big) \\ &> 4\big(2 \cdot 16 - 4 + 3x^2y^2\big) = 112 + 12x^2y^2 > 0 \quad \text{in } \Omega, \end{aligned}$$

so the PDE is hyperbolic.