10.1. PDE under change of coordinates For each of the following 3 questions, you have to provide a *numeric or symbolic answer*. An example of a numeric answer is $-27\sqrt{2}$; an example of a symbolic answer is a^3/b or $u_t = u_{xx}$. Insert your answers in the following grid. Write clearly so that there is no ambiguity. You do **not** have to provide any justification for your answers.

Question	1	2	3
Answer	$25v_{\eta\eta} = 16v_{\xi\xi}$	$8v_{\xi\xi} + 6v_{\xi\eta} + v_{\eta\eta} = 0$	$4v_{\xi\xi} + \frac{8\eta}{\xi}v_{\xi\eta} + (\frac{4\eta^2}{\xi^2} - \frac{\xi^2}{4})v_{\eta\eta} = 0$

Let $u(x,t)$ be a function satisfying $u_{xx} = u_{tt}$. For each of the following change of coordinates, write the PDE satisfied by write the PDE satisfied by $v(\xi,\eta) = u(\xi(x,t),\eta(x,t))$.

1. $\xi = 4x, \eta = 5t$.
2. $\xi = x + 3t, \eta = t$.
3. $\xi = 2x, \eta = xt$.

10.2. IVP under change of coordinates For each of the following 3 questions, you have to provide a *numeric or symbolic answer*. An example of a numeric answer is $-27\sqrt{2}$; an example of a symbolic answer is a^3/b or $u_t = u_{xx}$. Insert your answers in the following grid. Write clearly so that there is no ambiguity. You do **not** have to provide any justification for your answers.

Question	1	2	3
Answer	$\begin{cases} 25v_{\eta\eta} = 16v_{\xi\xi} \\ v(\xi,0) = f(\xi) \\ 5v_\eta(\xi,0) = g(\xi) \end{cases}$	$\begin{cases} 8v_{\xi\xi} + 6v_{\xi\eta} + v_{\eta\eta} = 0 \\ v(\xi,0) = f(\xi) \\ 3u_\xi(\xi,0) + v_\eta(\xi,0) = g(\xi) \end{cases}$	$\begin{cases} 4v_{\xi\xi} + \frac{8\eta}{\xi}v_{\xi\eta} + (\frac{4\eta^2}{\xi^2} - \frac{\xi^2}{4})v_{\eta\eta} = 0 \\ v(\xi,0) = f(\xi) \\ \frac{\xi}{2}v_\eta(\xi,0) = g(\xi) \end{cases}$

Consider the following IVP:

$$\begin{cases} u_{tt} = u_{xx} & \text{for } x \in \mathbb{R} \text{ and } t > 0, \\ u(x,0) = f(x) & \text{for } x \in \mathbb{R}, \\ u_t(x,0) = g(x) & \text{for } x \in \mathbb{R},\ t > 0. \end{cases}$$

For each of the following change of coordinates, write the corresponding IVP for $v(\xi,\eta) = u(\xi(x,t),\eta(x,t))$.

1. $\xi = 4x, \eta = 5t$.
2. $\xi = x + 3t, \eta = t$.
3. $\xi = 2x, \eta = xt$.

10.3. Vanishing mixed derivative

Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a twice differentiable function such that $u_{xy} = 0$ vanishes identically. Then show that there exists (twice differentiable) functions $a, b : \mathbb{R} \to \mathbb{R}$ such that

$$u(x, y) = a(x) + b(y), \quad \text{for all } (x, y) \in \mathbb{R}^2.$$

(This is the reason why it is convenient, e.g. in studying the 1D wave equation, to introduce the *canonical coordinates*).

Solution:

For notational convenience, let $(x_0, y_0) = (0, 0)$, the origin. The idea to solve this exercise is essentially as follows: given any (x, y) we take the equation $u_{xy} = 0$, first integrate horizontally from x_0 to x, and then vertically from y_0 to y. More precisely, for any $t \in \mathbb{R}$ we have that

$$u_y(x, t) - u_y(x_0, t) = \int_{x_0}^{x} u_{xy}(s, t)\, ds = 0,$$

thus

$$u_y(x, t) = u_y(x_0, t).$$

Hence, integrating both the left-hand side and the right-hand side of this equation for $y_0 \le t \le y$ we then get

$$u(x, y) - u(x, y_0) = u(x_0, y) - u(x_0, y_0),$$

which we can rewrite as

$$u(x, y) = u(x, y_0) + u(x_0, y) - u(x_0, y_0).$$

If we simply let $a(x) = u(x, y_0)$ and $b(y) = u(x_0, y) - u(x_0, y_0)$ the conclusion follows.

10.4. Pressure Wave

Consider the following equation:

$$\begin{cases} P_{tt} - P_{xx} = 0 & (x,t) \in \mathbb{R} \times \mathbb{R}_+ \\ P(x,0) = \begin{cases} 2, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases} \\ P_t(x,0) = \begin{cases} 1, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases} \end{cases}$$

(a) For which points in the domain does $P(x,t)$ automatically vanish? (Justify your answer with a drawing, without computing.)

(b) By using the d'Alembert's formula, compute $P(0,t)$ for all $t \geq 0$.

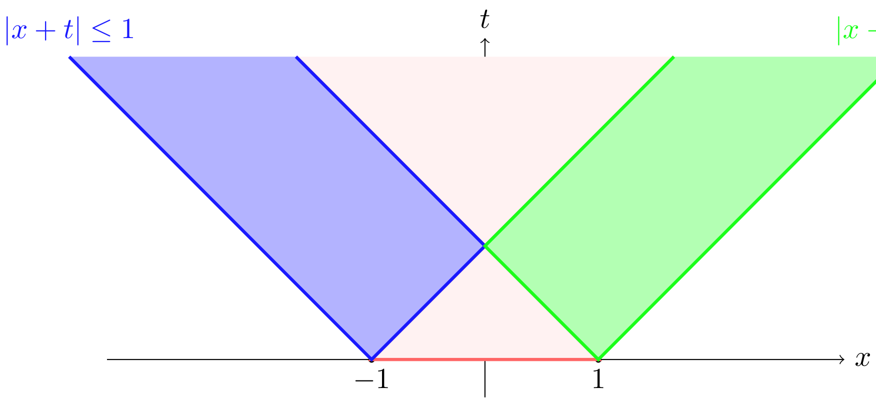
(c) *(Asymptotic behaviour for arbitrary x as $t \to \infty$).* Show that there is a constant $P_\infty \neq 0$ such that

$$\lim_{t\to\infty} P(x,t) = P_\infty$$

for every x in $\mathbb{R}$. That means that the "pressure" in the whole room is constant and not equal to zero after the sound has subsided.

Solution:

(a)



$P(x,t) = 0$ when $x + t < -1$ or $x - t > 1$.

(b) By d'Alembert's formula, we have

$$P(x,t) = \frac{1}{2}\left(f(x+t) - f(x-t) + \int_{x-t}^{x+t} g(s)ds\right),$$

where

$$f(s) = \begin{cases} 2, & |s| \leq 1 \\ 0, & |s| > 1 \end{cases} \qquad \text{and } g(s) = \begin{cases} 1, & |s| \leq 1 \\ 0, & |s| > 1 \end{cases}.$$

By the graph in **(a)**, we consider five cases:
Case 1. $x + t < -1$ or $x - t > 1$.

We have $f(x+t) = 0$, $f(x-t) = 0$, $g(s) = 0$ for all $s \in (x-t, x+t)$. Therefore

$$P(x,t) = 0.$$

Case 2. $x + t > 1$ and $x - t < -1$.

Similarly we have $f(x+t) = 0$, $f(x-t) = 0$. We compute $\int_{x-t}^{x+t} g(s)ds$:

$$\int_{x-t}^{x+t} g(s)ds = \int_{-1}^{1} g(s)ds = s|_{-1}^{1} = 2.$$

Hence

$$P(x,t) = \frac{1}{2}\int_{x-t}^{x+t} g(s)ds = 1.$$

Case 3. $|x + t| \leq 1$ and $|x - t| \leq -1$.

We have $f(x+t) = 2$ and $f(x-t) = 2$. Compute $\int_{x-t}^{x+t} g(s)ds$ and we have

$$\int_{x-t}^{x+t} g(s)ds = s|_{x-t}^{x+t} = 2t.$$

Hence

$$P(x,t) = 2 + t.$$

Case 4. $|x + t| \leq 1$ and $x - t < -1$.

We have $f(x+t) = 2$ and $f(x-t) = 0$. We compute

$$\int_{x-t}^{x+t} g(s)ds = \int_{-1}^{x+t} g(s)ds = s|_{-1}^{x+t} = x + t + 1.$$

Hence

$$P(x,t) = 1 + \frac{1}{2}(1 + x + t).$$

Case 5. $|x - t| \leq 1$ and $x + t > 1$.

We have $f(x+t) = 0$ and $f(x-t) = 2$. We compute

$$\int_{x-t}^{x+t} g(s)ds = \int_{x-t}^{1} g(s)ds = s|_{x-t}^{1} = 1 - x + t.$$

Hence

$$P(x,t) = 1 + \frac{1}{2}(1 - x + t).$$

Now we conclude:

$$P(x,t) = \begin{cases} 0, & x+t < -1 \text{ or } x-t > 1 \\ 1, & x+t > 1 \text{ and } x-t < -1 \\ 2+t, & |x+t| \leq 1 \text{ and } |x-t| \leq -1 \\ 1 + \frac{1}{2}(1+x+t), & |x+t| \leq 1 \text{ and } x-t < -1. \\ 1 + \frac{1}{2}(1-x+t), & |x-t| \leq 1 \text{ and } x+t > 1. \end{cases}$$

(c) We fix $x \in \mathbb{R}$. For large t we have $x+t > 1$ and $x-t < -1$, so we are in *Case 3.* from **(b)**. This gives $P_\infty = 1$ and we get

$$\lim_{t\to\infty} P(x,t) = P_\infty = 1.$$

10.5. Initial value problem on the real line Consider the following PDE:

$$\begin{cases} u_{tt} - 2u_{tx} = 24u_{xx} & \text{for } x \in \mathbb{R},\ t > 0, \\ u(x,0) = 0 & \text{for } x \in \mathbb{R}, \\ u_t(x,0) = \sin(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

(a) Let $v : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ be the function such that $v(x,t) = u(x-t,t)$ (or equivalently, consider the change of variables $(x',t') = (x-t,t)$). Compute the PDE satisfied by v.

(b) Using d'Alembert's formula, obtain a formula for v and deduce a formula for u.

Solution: Standard computations yield

$$\begin{aligned} \partial_x u &= \partial_x v, \\ \partial_t u &= \partial_t v + \partial_x v, \\ \partial_x u &= \partial_{xx} v, \\ \partial_{xt} u &= \partial_{xt} v + \partial_{xx} v, \\ \partial_{tt} u &= \partial_{tt} v + 2\partial_{xt} v + \partial_{xx} v. \end{aligned}$$

Therefore, we have

$$\begin{aligned}0 &= u_{tt} - 2u_{tx} - 24u_{xx} = \partial_{tt}v + 2\partial_{xt}v + \partial_{xx}v - 2\partial_{xt}v - 2\partial_{xx}v - 24v_{xx} = \partial_{tt}v - 25v_{xx}\\ \sin(x) &= \partial_t u(x,0) = \partial_t v(x,0) + \partial_x v(x,0) = \partial_t v(x,0).\end{aligned}$$

Thus the function v satisfies the wave equation (with propagation speed $c = 5$)

$$\begin{cases} v_{tt} - 25v_{xx} = 0 & \text{for } x \in \mathbb{R},\ t > 0,\\ v(x,0) = 0 & \text{for } x \in \mathbb{R},\\ v_t(x,0) = \sin(x) & \text{for } x \in \mathbb{R}.\end{cases}$$

Applying d'Alembert formula we obtain

$$v(x,t) = \frac{1}{10}\int_{x-5t}^{x+5t} \sin(\xi)d\xi = \frac{\cos(x-5t) - \cos(x+5t)}{10}.$$

Finally, since $u(x,t) = v(x+t,t)$, we get

$$u(x,t) = v(x+t,t) = \frac{\cos(x-4t) - \cos(x+6t)}{10}.$$