11.1. Eigenvalues of the rectangle For each of the following 5 statement you have to establish whether it is true or false.

Insert your answers in the following grid. Write clearly **T** if the statement is true and **F** if the statement is false. We will accept also **R** if the statement is *richtig* (which is the German word for *true*).

Only the answers in the grid will be taken into consideration for grading.

Question	1	2	3	4	5
Answer	F	T	F	T	F

Let $R := (0, a) \times (0, b)$ for $a, b > 0$. Let $\lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvalues (with multiplicity) of $-\Delta$ with Dirichlet boundary conditions on R, namely the values of $\lambda \in \mathbb{R}$ such that the following problem has a nontrivial solution

$$\begin{cases} -\Delta u = \lambda u & \text{in } R, \\ u = 0 & \text{on } \partial R. \end{cases}$$

1. There exists a negative eigenvalue.
2. If $a = 2\pi$ and $b = 5\pi$, then $\lambda_1 = \frac{29}{100}$.
3. If $a = b = \pi$, the multiplicity of 65 as eigenvalue is 2.
4. If $a = 5\pi, b = 2021$, there is not an *integer* eigenvalue.
5. If $a = b = 1$, then $\lambda_{2021} \leq 100$.

Solution: You have seen during the lecture that all the eigenvalues are given by

$$\frac{x^2\pi^2}{a^2} + \frac{y^2\pi^2}{b^2} \tag{1}$$

where x, y are positive integer numbers.

1. Since any eigenvalue is the sum of two squares, it cannot be negative.
2. The smallest eigenvalue is given by the expression (1) with $x = y = 1$, so

$$\lambda_1 = \frac{\pi^2}{4\pi^2} + \frac{\pi^2}{25\pi^2} = \frac{29}{100}.$$

3. Thanks to the expression (1), the multiplicity of 65 is given by the number of pairs (x, y) of positive integers such that
$$x^2 + y^2 = 65.$$
One can check that there are exactly 4 such pairs $(8,1),(1,8),(7,4),(4,7)$, so the multiplicity of 65 as eigenvalue is 4.

4. We will show that no eigenvalue is integer. Assume by contradiction that for some positive integers x, y, the corresponding eigenvalue is an integer number n, so we have
$$\frac{x^2}{25} + \frac{y^2\pi^2}{2021^2} = n.$$
Hence we deduce that π^2 is a rational number:
$$\pi^2 = \frac{2021^2(n - \frac{x^2}{25})}{y^2},$$
but it is well known that π is not a rational number so this is impossible and yields a contradiction.

5. Notice that $\lambda_{2021} \leq 100$ if and only if there are at least 2021 eigenvalues below 100 (counted with multiplicities). Thanks to the expression (1), this is equivalent to checking whether there are 2021 distinct pairs of positive integers (x, y) such that
$$x^2 + y^2 \leq \frac{100}{\pi^2}.$$
If the latter inequality holds, then $1 \leq x, y \leq \frac{10}{\pi}$ and therefore there are at most $\left(\frac{10}{\pi}\right)^2 \leq 4^2 = 16$ eigenvalues below 100, so $\lambda_{2021} > 100$.

11.2. Laplace equation in the square Let $R := (0,1) \times (0,1) \subset \mathbb{R}^2$. Compute the solution $u : R \to \mathbb{R}$ of the following Dirichlet problem
$$\begin{cases} \Delta u(x,y) = 0 & \text{in } R \\ u(x,y) = f(x,y) & \text{on } \partial R, \end{cases}$$
where

(a)
$$f(x,y) = \begin{cases} 0 & \text{for} \quad y = 0\,, \\ 0 & \text{for} \quad y = 1\,, \\ 0 & \text{for} \quad x = 0\,, \\ -\sin(2\pi y)\cos(2\pi y) & \text{for} \quad x = 1. \end{cases}$$

(b)

$$f(x,y) = \begin{cases} x(x-1) & \text{for} \quad y = 0\,, \\ 0 & \text{for} \quad y = 1\,, \\ 0 & \text{for} \quad x = 0\,, \\ 0 & \text{for} \quad x = 1. \end{cases}$$

Hint: Write u as a sum of functions of the form $X_n(x)Y_n(y)$.

Solution:

(a) We use the separation Ansatz $u(x,y) = X(x)Y(y)$. The PDE then becomes

$$\frac{X''}{X} = -\frac{Y''}{Y} =: \lambda$$

with the boundary condition $Y(0) = 0 = Y(1)$ and $X(0) = 0$. The Y-problem

$$\begin{cases} Y'' + \lambda Y = 0 \\ \quad Y(0) = 0 \\ \quad Y(1) = 0 \end{cases}$$

has nontrivial solutions for $\lambda = (\pi n)^2$, $n \in \mathbb{N}_{\geq 1}$. Then we get

$$Y_n(y) = A_n \sin(\pi n y), \quad A_n \in \mathbb{R}.$$

For $\lambda = (\pi n)^2$, the Y-problem

$$\begin{cases} X'' - \lambda X = 0 \\ \quad X(0) = 0 \end{cases}$$

has the nontrivial solution

$$X_n(x) = B_n \mathrm{e}^{\pi n x} - B_n \mathrm{e}^{-\pi n x} = C_n \sinh(\pi n x), \quad C_n \in \mathbb{R}.$$

By superposition, we have

$$u(x,y) = \sum_{n=1}^{\infty} X_n(x) Y_n(y) = \sum_{n=1}^{\infty} D_n \sin(\pi n y) \sinh(\pi n x).$$

The constants D_n are determined by the inhomogeneous boundary condition. We have

$$u(1,y) = \sum_{n=1}^{\infty} D_n \sinh(\pi n) \sin(n\pi y) = -\sin(2\pi y)\cos(2\pi y).$$

Since $-\sin(2\pi y)\cos(2\pi y) = \frac{-\sin(4\pi y)}{2}$, we have

$$u(x,y) = -\frac{\sin(4\pi y)\sinh(4\pi x)}{2\sinh(4\pi)}.$$

(b) With the separation Ansatz $u(x,y) = X(x)Y(y)$, we have

$$
\begin{aligned}
\frac{X''(x)}{X(x)} &= -\frac{Y''(y)}{Y(y)} = \text{const.} = \lambda,\\
X(x)Y(0) &= x(x-1),\\
X(x)Y(1) &= 0 \Longrightarrow Y(1) = 0,\\
X(0)Y(y) &= 0 \Longrightarrow X(0) = 0,\\
X(1)Y(y) &= 0 \Longrightarrow X(1) = 0.
\end{aligned}
$$

For $\lambda = -\pi^2 k^2$, $k \in \mathbb{N}_{\geq 1}$, the problem

$$
\begin{cases}
X''(x) - \lambda X(x) = 0,\\
X(0) = 0,\\
X(1) = 0,
\end{cases}
$$

has nontrivial solution

$$X(x) = X_k(x) = A_k \sin(k\pi x) \quad x \in [0,1], \text{ for any } A_k \in \mathbb{R}$$

For $\lambda = -\pi^2 k^2$, $k \in \mathbb{N}_{\geq 1}$, the solution of

$$
\begin{cases}
Y''(y) - \pi^2 k^2 Y(y) = 0,\\
Y(1) = 0,
\end{cases}
$$

is $Y(y) = Y_k(y) = B_k \mathrm{e}^{-\pi k y}(e^{2\pi k} - \mathrm{e}^{2\pi k y})$, $B_k \in \mathbb{R}$. By the superposition principle, we have

$$u(x,y) = \sum_{k=1}^{\infty} C_k \mathrm{e}^{-\pi k y}(e^{2\pi k} - \mathrm{e}^{2\pi k y}) \sin(\pi k x).$$

To compute C_k, we have

$$u(x,0) = \sum_{k=1}^{\infty} (e^{2\pi k} - 1) C_k \sin(k\pi x) \stackrel{!}{=} x(x-1)$$

and hence

$$(e^{2\pi k} - 1) C_k = 2\int_0^1 x(x-1)\sin(\pi k x)\,\mathrm{d}x = \frac{4((-1)^k - 1)}{k^3 \pi^3};$$

therefore, u is given by

$$u(x,y) = \sum_{k=1}^{\infty} \frac{4((-1)^k - 1)}{k^3\pi^3} \mathrm{e}^{-\pi k y} \frac{e^{2\pi k} - \mathrm{e}^{2\pi k y}}{e^{2\pi k} - 1} \sin(\pi k x).$$

11.3. Laplace equation with mixed boundary conditions Compute the solution $u : R \to \mathbb{R}$ of the following boundary value problem

$$\begin{cases} \Delta u(x,y) = 0 & \text{for } (x,y) \in (0,1)^2, \\ u(x,y) = 0 & \text{for } (x,y) \in \{0,1\} \times (0,1) \cup (0,1) \times \{0\}, \\ u + u_y = \sin(\pi x) & \text{for } (x,y) \in (0,1) \times \{1\}. \end{cases}$$

Solution: Ignoring the last boundary condition (the one for $y = 1$), we can repeat the same argument we employed for **(a)** in the previous exercise (up to swapping x and y) and deduce that the general solution to the problem is given by

$$u(x,y) = \sum_{n \in \mathbb{N}} a_n \sin(n\pi x) \sinh(n\pi y).$$

It remains to determine the coefficients a_n, in order to do so we shall impose the validity of the missing boundary condition:

$$\sin(\pi x) = u(x,1) + u_y(x,1) = \sum_{n \in \mathbb{N}} a_n \sin(n\pi x)(\sinh(n\pi) + n\pi \cosh(n\pi)).$$

Hence, we deduce that $a_n = 0$ for any $n \geq 2$ and $1 = a_1(\sinh(\pi) + \pi \cosh(\pi))$. Therefore the solution to the problem is

$$u(x,y) = \frac{\sin(\pi x) \sinh(\pi y)}{\sinh(\pi) + \pi \cosh(\pi)}.$$