12.1. Laplace equation For each of the following 5 statement you have to establish whether it is true or false.

Insert your answers in the following grid. Write clearly **T** if the statement is true and **F** if the statement is false. We will accept also **R** if the statement is *richtig* (which is the German word for *true*).

Only the answers in the grid will be taken into consideration for grading.

Question	1	2	3	4	5
Answer	F	T	T	T	F

In this exercise, $\Delta = \partial_{xx} + \partial_{yy}$ is the Laplace operator in $\mathbb{R}^2$.

1. If $\Delta u = 0$ and $\Delta v = 0$ for all $(x, y) \in \mathbb{R}^2$, then for any real-valued smooth functions $c_1, c_2 : \mathbb{R}^2 \to \mathbb{R}$, we have
$$\Delta(c_1 u + c_2 v) = 0.$$
2. If $\Delta u = 0$ in $\mathbb{R}^2$, then $\Delta(\partial_x u) = 0$ in $\mathbb{R}^2$.
3. Let $D := \{(x, y) : x^2 + y^2 < 4\}$. There exist infinitely many functions $u : D \to \mathbb{R}$ such that
$$\begin{cases} \Delta u = 5 & \text{in } D, \\ u(1, 1) = 0. \end{cases}$$
4. Let $D := \{(x, y) : x^2 + y^2 = 1\}$. If $u : D \to \mathbb{R}$ solves
$$\begin{cases} \Delta u = 0 & \text{in } D, \\ u(x, y) = 2 - x^3 & \text{for } (x, y) \in \partial D, \end{cases}$$
then $u(0, \frac{1}{2}) > 1$.
5. Let $D := \{(x, y) : x^2 + y^2 = 1\}$. If $u : D \to \mathbb{R}$ solves
$$\begin{cases} \Delta u = 0 & \text{in } D, \\ u(x, y) = \sin(x) & \text{for } (x, y) \in \partial D, \end{cases}$$
then $u(0, 0) = \frac{1}{\pi}$.

Solution:

1. If we set $u = 1, v = 0, c_1 = x^2, c_2 = 0$, we would have that $\Delta u = \Delta v = 0$, but $\Delta(c_1 u + c2v) = \Delta(x^2) = 2 \neq 0$.

2. Since the derivatives commute one with the other, we have $\Delta(\partial_x u) = \partial_x(\Delta u) = \partial_x(0) = 0$.

3. Intuitively there are infinitely many solutions because instead of giving boundary conditions we are giving the value only at one point. One way to construct infinitely many solutions is the following. Given a positive integer $k \geq 1$, let

$$u(x, y) := \mathrm{Re}\Big(\big((x + iy) - (1 + i)\big)^k\Big) + \frac{5}{4}\big((x - 1)^2 + (y - 1)^2\big) + 1.$$

The first term is harmonic (recall that $\mathrm{Re}(z^k)$ is harmonic when $z := x + iy$) and is null at $(1, 1)$. The second term has Laplacian equal to 5 and is 0 at $(1, 1)$. The third term is harmonic and its value at $(1, 1)$ is 1. Hence, the function u we defined satisfies all the conditions on the whole $\mathbb{R}^2$ (and not only on D).

4. Notice that on the boundary $|x| \leq 1$, thus $u(x, y) = 2 - x^3 \geq 2 - 1 \geq 1$ for all $(x, y) \in \partial D$. Thus, by the maximum principle $u(x, y) > 1$ for all $(x, y) \in D$.

5. By the mean property for harmonic functions,

$$u(0, 0) = \frac{1}{2\pi}\int_0^{2\pi} u(\cos(\theta), \sin(\theta))\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \sin(\cos(\theta))\, d\theta = 0.$$

12.2. Laplace equation in a disk Let $D := \{(x, y) : x^2 + y^2 < 1\}$. We want to solve the Dirichlet boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ u(x, y) = 1 + 3x^4 & \text{for } (x, y) \in \partial D. \end{cases}$$

(a) Prove $1 \leq u(x, y) \leq 4$ for all $(x, y) \in D$.

(b) Compute $u(0, 0)$ with the Poisson formula.

(c) Compute u explicitly at all points $(x, y) \in D$ and prove $1 \leq u(x, y) \leq 4$ using this explicit expression.

Solution:

(a) By the maximum principle, the maximum and minimum of u in $\overline{D}$ are achieved on ∂D. For $(x, y) \in \partial D$, we have

$$u(x, y) = 1 + 3x^4 \leq 1 + 3 \cdot 1 = 4$$

because $|x| \leq 1$ for $x \in \partial D$ and

$$u(x, y) = 1 + 3x^4 \geq 1.$$

(b) According to the mean value property of harmonic functions (this is just the fact that the Poisson kernel with $r = 0$ is the constant function 1):

$$\begin{aligned} u(0,0) &= \frac{1}{2\pi} \int_0^{2\pi} u(\cos(\theta), \sin(\theta))\ d\theta = \frac{1}{2\pi} \int_0^{2\pi} \left(1 + 3\cos^4(\theta)\right)\ d\theta \\ &= 1 + \frac{3}{2\pi} \underbrace{\int_0^{2\pi} \cos^4(\theta)\ d\theta}_{=:I}. \end{aligned}$$

Now we compute I:

$$\begin{aligned} I &= \int_0^{2\pi} \cos^4(x)\ dx = \underbrace{\sin(x)\cos^3(x)\Big|_{x=0}^{x=2\pi}}_{=0} - \int_0^{2\pi} \sin(x) 3\cos^2(x)(-\sin(x))\ dx \\ &= \frac{3}{4} \int_0^{2\pi} \sin^2(2x)\ dx = \frac{3}{8} \int_0^{4\pi} \sin^2(y)\ dy = \frac{3}{8} \cdot \frac{4\pi}{2} = \frac{3}{4}\pi, \end{aligned}$$

then we have

$$u(0,0) = 1 + \frac{3}{2\pi} \cdot \frac{3}{4}\pi = \frac{17}{8}.$$

(c) Finally, we compute the function u using the Fourier series. We know from the lecture that if (in polar coordinates)

$$u(1, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right),$$

then

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right).$$

Note that

$$\cos^4(\theta) = \frac{3}{8} + \frac{\cos(2\theta)}{2} + \frac{\cos(4\theta)}{8}$$

and then

$$1 + 3\cos^4(\theta) = 1 + 3(\frac{3}{8} + \frac{\cos(2\theta)}{2} + \frac{\cos(4\theta)}{8}) = \frac{17}{8} + \frac{3}{2}\cos(2\theta) + \frac{3}{8}\cos(4\theta).$$

Now we have the Fourier series of $u(1, \cdot)$ and it gives[1]

$$u(r, \theta) = \frac{17}{8} + r^2\frac{3}{2}\cos(2\theta) + \frac{3}{8}r^4\cos(4\theta).$$

An elementary computation shows that u solves the boundary value problem.

From this explicit expression we have $0 \leq r < 1$ und $-1 \leq \cos(2\theta) \leq 1$ and $-1 \leq \cos(4\theta) \leq 1$ for any $\theta \in \mathbb{R}$:

$$u(r, \theta) \leq \frac{17}{8} + \frac{3}{2} + \frac{3}{8} = 4.$$

Also we have

$$\begin{aligned}
u(r, \theta) &= \frac{17}{8} + \frac{3}{2}r^2\cos(2\theta) + \frac{3}{8}r^4\cos(4\theta) \\
&= \frac{17}{8} + \frac{3}{2}r^2\cos(2\theta) + \frac{3}{8}r^4\left(2\cos^2(2\theta) - 1\right) \\
&= \frac{14}{8} + \frac{3}{2}r^2\cos(2\theta) + \frac{3}{4}\left(r^2\cos(2\theta)\right)^2 \\
&= \frac{14}{8} + \frac{3}{4}\left(\left(r^2\cos(2\theta) + 1\right)^2 - 1\right) \\
&\geq \frac{14}{8} - \frac{3}{4} = 1.
\end{aligned}$$

This concludes our proof.

12.3. Laplace equation in an annulus Let $D := \{(x, y) : 1 < x^2 + y^2 < 4\}$. Find the solution $u : D \to \mathbb{R}$ of the following problem

$$\begin{cases} \Delta u = 0 \quad \text{in } D, \\ u(1, \theta) = 2\sin(2\theta) \quad \text{for } 0 \leq \theta < 2\pi, \\ u(2, \theta) = 3\cos(3\theta) \quad \text{for } 0 \leq \theta < 2\pi. \end{cases}$$

[1] In Cartesian coordinate (x, y), we have

$$\begin{aligned}
u(x, y) &= \frac{17}{8} + \frac{3}{2}r^2\cos(2\theta) + \frac{3}{8}r^4\cos(4\theta) \\
&= \frac{17}{8} + \frac{3}{2}\left(r^2\cos^2(\theta) - r^2\sin^2(\theta)\right) + \frac{3}{8}r^4\left(\cos^2(2\theta) - \sin^2(2\theta)\right) \\
&= \frac{17}{8} + \frac{3}{2}(x^2 - y^2) + \frac{3}{8}r^4\left(\left(\cos^2(\theta) - \sin^2(\theta)\right)^2 - 4\sin^2(\theta)\cos^2(\theta)\right) \\
&= \frac{17}{8} + \frac{3}{2}(x^2 - y^2) + \frac{3}{8}\left((x^2 - y^2)^2 - 4x^2y^2\right)
\end{aligned}$$

Solution: Based on what has been discussed in class (Lecture 12), we know that the general solution of the PDE (without accounting for the boundary conditions) takes the form

$$u(r, \theta) = \sum_{n \in \mathbb{Z}} r^n (a_n \cos(n\theta) + b_n \sin(n\theta)).$$

The first boundary condition implies

$$2 \sin(2\theta) = \sum_{n \in \mathbb{Z}} a_n \cos(n\theta) + b_n \sin(n\theta),$$

thus we have:

- $a_0 = b_0 = 0$.
- $a_n + a_{-n} = 0$ for all $n \geq 1$,
- $b_n - b_{-n} = 0$ for $n = 1$ and $n > 2$,
- $b_2 - b_{-2} = 2$.

The second boundary condition implies

$$3 \cos(3\theta) = \sum_{n \in \mathbb{Z}} 2^n (a_n \cos(n\theta) + b_n \sin(n\theta)),$$

thus we have:

- $a_0 = b_0 = 0$.
- $2^n b_n - 2^{-n} b_{-n} = 0$ for all $n \geq 1$,
- $2^n a_n + 2^n a_{-n} = 0$ for $n \in \{1, 2\}$ and $n > 3$,
- $8a_3 + \frac{1}{8} a_{-3} = 3$.

Joining the information we have, we deduce

- $a_n = 0$ for all $n \neq 3, -3$,
- $b_n = 0$ for all $n \neq 2, -2$,
- $$\begin{cases} 8a_3 + \frac{1}{8} a_{-3} = 3 \\ a_3 + a_{-3} = 0 \end{cases}$$

- $$\begin{cases} b_2 - b_{-2} = 2 \\ 4b_2 - \frac{1}{4}b_{-2} = 0 \end{cases}$$

Solving the linear systems one gets $a_3 = \frac{24}{63}, a_{-3} = \frac{-24}{63}, b_2 = \frac{-2}{15}, b_{-2} = \frac{-32}{15}$. Therefore the solution u is given by

$$u(r,\theta) = \frac{24}{63}(r^3 - r^{-3})\cos(3\theta) + \frac{1}{15}(-2r^2 + 32r^{-2})\sin(2\theta).$$