2.1. Polar coordinates Every function $u : \mathbb{R}^2 \to \mathbb{R}$, $u = u(x, y)$ can be expressed in polar coordinates, i.e.

$$\begin{cases} x = x(r, \theta) = r\cos(\theta) \\ y = x(r, \theta) = r\sin(\theta) \end{cases}$$

then we write $u(r, \theta) = u(x(r, \theta), y(r, \theta))$.

(a) Rewrite the following function from Cartesian coordinates to polar coordinates

$$u_1(x, y) = x^2 + y^2, \qquad u_2(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \qquad u_3(x, y) = \frac{x}{y}.$$

(b) Rewrite the following function from polar coordinates to Cartesian coordinates

$$v_1(r, \theta) = r^n, \qquad v_2(r, \theta) = \sin\theta\cos\theta, \qquad v_3(r, \theta) = \theta \quad (-\pi/2 < \theta < \pi/2),$$
$$v_4(r, \theta) = \theta \quad (\pi/2 < \theta < 3\pi/2).$$

Solution:

(a) $u_1(r, \theta) = r^2$, $u_2(r, \theta) = \cos^2\theta - \sin^2\theta$, $u_3(r, \theta) = \cot\theta$.

(b)

$$v_1(x, y) = \left(x^2 + y^2\right)^{n/2},$$

$$v_2(x, y) = \frac{xy}{x^2 + y^2},$$

$$v_3(x, y) = \arcsin\left(\frac{y}{\sqrt{x^2 + y^2}}\right),$$

$$v_4(x, y) = -\arcsin\left(\frac{y}{\sqrt{x^2 + y^2}}\right) + \pi.$$

2.2. The Laplace operator in polar coordinates

Using the same notation to denote polar coordinates introduced in the previous exercise, show the following statements.

(a) Use chain rule to prove

$$\partial_r u(r,\theta) = (\partial_x u)\cos\theta + (\partial_y u)\sin\theta$$
$$\partial_\theta u(r,\theta) = -(\partial_x u) r\sin\theta + (\partial_y u) r\cos\theta.$$

Now we have the following relation for the partial derivatives $\partial_x u$ and $\partial_y u$:

$$\begin{pmatrix}\partial_r u\\ \partial_\theta u\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -r\sin\theta & r\cos\theta\end{pmatrix}\begin{pmatrix}\partial_x u\\ \partial_y u\end{pmatrix}.$$

(b) By inverting some matrix, prove the following expressions for $\partial_x u$ and $\partial_y u$:

$$\partial_x u = \cos\theta(\partial_r u) - \frac{1}{r}\sin\theta(\partial_\theta u),$$
$$\partial_y u = \sin\theta(\partial_r u) + \frac{1}{r}\cos\theta(\partial_\theta u).$$

Use these formulas and chain rule to compute the direct expressions for $\partial^2_{xx} u$ and $\partial^2_{yy} u$ in polar coordinates, i.e.

$$\partial^2_{xx} u = \partial_x(\partial_x u) = \cos\theta(\partial_r(\partial_x u)) - \frac{1}{r}\sin\theta(\partial_\theta(\partial_x u)) = ...$$

(c) Combine all the information above and prove the following expression for the Laplacian operator in polar coordinates

$$\boxed{\Delta u(r,\theta) = \partial^2_{rr} u + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial^2_{\theta\theta} u.}$$

Solution:

(a) Applying the chain rule for the derivative, we have

$$\partial_r u(r,\theta) = \partial_r u(r\cos(\theta), r\sin(\theta)) = \partial_x u \cdot \partial_r(r\cos\theta) + \partial_y u \cdot \partial_r(r\sin\theta) = \partial_x u\cos\theta + \partial_y u\sin\theta\,.$$

Similarly, we have

$$\partial_\theta u(r,\theta) = \partial_\theta u(r\cos(\theta), r\sin(\theta)) = \partial_x u \cdot \partial_\theta(r\cos\theta) + \partial_y u \cdot \partial_\theta(r\sin\theta) = -r\sin\theta\partial_x u + r\cos\theta\partial_y u\,.$$

(b) The inverse of this matrix

$$A = \begin{pmatrix}\cos\phi & \sin\phi\\ -r\sin\phi & r\cos\phi\end{pmatrix}$$

is

$$A^{-1} = \frac{1}{r}\begin{pmatrix} r\cos\phi & -\sin\phi \\ r\sin\phi & \cos\phi \end{pmatrix}.$$

Then we have

$$A^{-1}\begin{pmatrix} \partial_r u \\ \partial_\phi u \end{pmatrix} = \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix} \iff \frac{1}{r}\begin{pmatrix} r\cos\phi & -\sin\phi \\ r\sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} \partial_r u \\ \partial_\phi u \end{pmatrix} = \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix}.$$

This gives

$$\begin{aligned} \partial_x u &= \cos\phi\partial_r u - \frac{1}{r}\sin\phi\partial_\phi u, \\ \partial_y u &= \sin\phi\partial_r u + \frac{1}{r}\cos\phi\partial_\phi u. \end{aligned}$$

Now we compute $\partial^2_{xx}u$ and $\partial^2_{yy}u$,

$$\partial^2_{xx}u = \partial_x(\partial_x u) = \cos\phi(\partial_r(\partial_x u)) - \frac{1}{r}\sin\phi(\partial_\phi(\partial_x u)),$$

$$\partial_r(\partial_x u) = \cos\phi\partial^2_{rr}u + \frac{1}{r^2}\sin\phi\partial_\phi u - \frac{1}{r}\sin\phi\partial^2_{r\phi}u,$$

and

$$\partial_\phi(\partial_x u) = -\sin\phi\partial_r u + \cos\phi\partial^2_{r\phi}u - \frac{1}{r}\cos\phi\partial_\phi u - \frac{1}{r}\sin\phi\partial^2_{\phi\phi}u.$$

This gives

$$\partial^2_{xx}u = \partial^2_{rr}u\cos^2\phi + \partial^2_{\phi\phi}u\left(\frac{1}{r^2}\sin^2\phi\right) + \partial^2_{r\phi}u\left(-\frac{2}{r}\sin\phi\cos\phi\right) + \partial_r u\frac{1}{r}\sin^2\phi + \partial_\phi u\frac{2}{r^2}\sin\phi\cos\phi.$$

Similarly,

$$\partial^2_{yy}u = \partial_y(\partial_y u) = \sin\phi(\partial_r(\partial_y u)) + \frac{1}{r}\cos\phi(\partial_\phi(\partial_y u)).$$

$$\partial_r(\partial_y u) = \sin\phi\partial^2_{rr}u - \frac{1}{r^2}\cos\phi\partial_\phi u + \frac{1}{r}\cos\phi\partial^2_{r\phi}u,$$

and

$$\partial_\phi(\partial_y u) = \cos\phi\partial_r u + \sin\phi\partial^2_{r\phi}u - \frac{1}{r}\sin\phi\partial_\phi u + \frac{1}{r}\cos\phi\partial^2_{\phi\phi}u.$$

Then we have

$$\partial^2_{yy}u = \partial^2_{rr}u\sin^2\phi + \partial^2_{\phi\phi}u\left(\frac{1}{r^2}\cos^2\phi\right) + \partial^2_{r\phi}u\left(\frac{2}{r}\sin\phi\cos\phi\right) + \partial_r u\frac{1}{r}\cos^2\phi - \partial_\phi u\frac{2}{r^2}\sin\phi\cos\phi.$$

(c) From **(b)** and $\sin^2\phi + \cos^2\phi = 1$ we have

$$\Delta u = \partial^2_{rr}u + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial^2_{\phi\phi}u.$$

2.3. The heat equation on a thin disk Consider a thin (homogeneous) metal disk of radius $r_0 > 0$, whose temperature profile we shall describe in polar coordinates by means of a function $u(r,\theta,t)$. The initial temperature, at time t_0, is a known function $u_0 = u_0(r,\theta)$ and the body is completely insulated.

(a) Write down the full initial boundary value problem (IBVP) modelling the situation described above.

Tip: what is the exterior unit normal to a disk?

(b) What is the solution of this problem in the special case when the initial temperature is constant (equal to T_0)?

(c) Consider the problem in the special case when the initial temperature is a purely radial function, i.e. $u_0(r,\theta) = v_0(r)$. Make the ansatz that also the solution $u(r,\theta,t)$ does not depend on θ, i.e., $u(r,\theta,t) = v(r,t)$. Write down the equations satisfied by v. Prove that the quantity

$$\int_0^{r_0} r\, v(r,t)\, dr$$

does not depend on t. What is the phisical meaning of such quantity? Can you find the asymptotic state of this solution? The asymptotic state is the limit function $v_\infty(r) := \lim_{t\to\infty} v(r,t)$ and can be obtained by coupling the equations satisfied by v with the additional requirement $\partial_t v = 0$.

Solution:

(a) The initial condition is

$$u(r,\theta,t_0) = u_0(r,\theta) \quad \text{for } 0 \le r \le r_0, 0 \le \theta < 2\pi\,.$$

The equation satisfied by u is the heat equation (we use the expression for the Laplacian obtained in the previous exercise)

$$\partial_t u = \partial^2_{rr}u + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial^2_{\theta\theta}u \quad \text{for all } 0 \le r < r_0,\ 0 \le \theta < 2\pi,\ t > t_0\,.$$

Finally, the boundary conditions are null Neumann boundary conditions because the body is completely insulated (notice that the normal derivative coincides with ∂_r)

$$\partial_r u(r_0,\theta,t) = 0 \quad \text{for } 0 \le \theta < 2\pi,\ t > t_0\,.$$

(b) One would expect that if the body is initially at temperature T_0, it remains at such temperature. In fact, this is the case. One can check that the function $u \equiv T_0$ solves the initial boundary value problem described in **(a)**.

(c) Adapting the equations stated in **(a)** to the radial case, we get

$$\begin{cases} v(r, t_0) = v_0(r) \quad \text{for } 0 \leq r \leq r_0\,. \\ \partial_t v = \partial^2_{rr} v + \frac{1}{r}\partial_r v \quad \text{for all } 0 \leq r < r_0,\, t > t_0\,, \\ \partial_r v(r_0, t) = 0 \quad \text{for } t > t_0\,. \end{cases}$$

Notice that, by integrating in polar coordinates ($x = (r\cos\theta, r\sin\theta)$), one gets (the set B_{r_0} represents the disk with radius r_0)

$$\int_{B_{r_0}} u(x,t)\,dx = 2\pi \int_0^{r_0} r\,v(r,t)\,dr\,,$$

thus, up to a constant multiplicative coefficient, the quantity $\int_0^{r_0} r\,v(r,t)\,dr$ represents the average temperature of the body. Since the body is completely insulated, we expect its average temperature to remain constant. Let us show it by computing its derivative (we are going to use the equations satisfied by v)

$$\frac{d}{dt}\int_0^{r_0} r\,v(r,t)\,dr = \int_0^{r_0} r\,\partial_t v(r,t)\,dr = \int_0^{r_0} r\,\partial^2_{rr} v + \partial_r v\,dr = \int_0^{r_0} \partial_r\left(r\partial_r v\right)dr = \Big[r\partial_r v\Big]_0^{r_0} = 0\,.$$

Finally, let us investigate the asymptotic state v_∞. Since it is obtained as a limit of v when $t \to \infty$, and the quantity $\int_0^{r_0} r\partial_r v$ does not depend on time, it must hold

$$\int_0^{r_0} r\,v_\infty(r)\,dr = \int_0^{r_0} r\,v_0(r)\,dr\,. \tag{1}$$

Furthermore, v_∞ satisfies the same equations of v but does not depend on time, thus

$$\begin{cases} 0 = \partial^2_{rr} v_\infty + \frac{1}{r}\partial_r v_\infty \quad \text{for all } 0 \leq r < r_0\,, \\ \partial_r v_\infty(r_0) = 0\,. \end{cases}$$

Notice that $\partial^2_{rr} v_\infty + \frac{1}{r}\partial_r v_\infty = \frac{1}{r}\partial_r(r\partial_r v_\infty)$, thus we have

$$\begin{cases} \partial_r(r\partial_r v_\infty) = 0 \quad \text{for all } 0 \leq r < r_0\,, \\ (r\partial_r v_\infty)(r_0) = 0\,. \end{cases}$$

Since a function with null derivative is constant, we deduce that $\partial_r v_\infty(r) = 0$ for all $0 \leq r < r_0$ and therefore v_∞ is a constant function. So, we obtain from (1) that

$$v_\infty \equiv \frac{2}{r_0^2}\int_0^{r_0} r\,v_0(r)\,dr\,.$$